

On the extended multi-component Toda hierarchy

Chuanzhong Li[†], Jingsong He^{‡*}

Department of Mathematics, Ningbo University, Ningbo, 315211, China

[†]lichuanzhong@nbu.edu.cn

[‡]hejingsong@nbu.edu.cn

Abstract

The extended flow equations of the multi-component Toda hierarchy are constructed. We give the Hirota bilinear equations and tau function of this new extended multi-component Toda hierarchy (EMTH). Because of logarithmic terms, some extended vertex operators are constructed in generalized Hirota bilinear equations which might be useful in topological field theory and Gromov-Witten theory. Meanwhile the Darboux transformation and bi-Hamiltonian structure of this hierarchy are given. From the Hamiltonian tau symmetry, we give another different tau function of this hierarchy with some unknown mysterious connections with the one defined from the point of wave functions.

Mathematics Subject Classifications(2000): 37K05, 37K10, 37K40.

Keywords: multi-component Toda hierarchy, extended multi-component Toda hierarchy, Darboux transformation, bi-Hamiltonian structure, tau function

1 Introduction

The KP hierarchy and Toda lattice hierarchy as completely integrable systems have many important applications in mathematics and physics including the representation theory of Lie algebra, orthogonal polynomials and random matrix model [1–5]. The KP and Toda systems have many kinds of reductions or extensions, for example the BKP, CKP hierarchy, extended Toda hierarchy (ETH) [6, 7], bigraded Toda hierarchy (BTH) [8]–[14] and so on [15]. There are some other generalizations called multi-component KP [16, 17] and multi-component Toda systems [18] which attract more and more attention because their widely use in many fields such as the fields of multiple orthogonal polynomials and non-intersecting Brownian motions.

The multicomponent KP hierarchy was discussed with its application in representation theory and random matrix model in [16, 17]. In [3], it was noticed that the τ functions of a $2N$ -multicomponent KP provide solutions of the N -multicomponent 2D Toda hierarchy. The multicomponent 2D Toda hierarchy was considered from the point of view of the Gauss-Borel factorization problem, the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions and matrix Riemann-Hilbert problem [18]–[21]. In fact

*corresponding author

the multicomponent 2D Toda hierarchy in [19] is a periodic reduction of the bi-infinite matrix-formed two dimensional Toda hierarchy. The coefficients(or dynamic variables) of the multicomponent 2D Toda hierarchy take values in complex finite-sized matrices. The multicomponent 2D Toda hierarchy contains the matrix-formed Toda equation as the first flow equation.

Considering its application in the Gromov-Witten theory, the Toda hierarchy was extended to the extended Toda hierarchy [6] which governs the Gromov-Witten invariants of CP^1 . The extended bigraded Toda hierarchy(EBTH) is the extension of the bigraded Toda hierarchy (BTH) which includes $N + M$ series of additional logarithmic flows [8, 10] with considering its application in the Gromov-Witten theory of orbifolds $C_{N,M}$. The Hirota bilinear equations of EBTH were equivalently constructed in our early paper [9] and a very recent paper [22], because the equivalence of $t_{1,N}$ flow and $t_{0,N}$ flow of EBTH. Meanwhile it was proved to govern Gromov-Witten invariants of the total descendent potential of \mathbb{P}^1 orbifolds $C_{N,M}$ [22]. A natural question is what about the corresponding extended multi-component Toda hierarchy(as a matrix-valued generalization of extended Toda hierarchy [6]) and extended multicomponent bigraded Toda hierarchy. There is a class of orbifolds which should be governed by these generalized multicomponent logarithmic hierarchies. That is why we think this new kind of logarithmic hierarchy which might be useful in Gromov-Witten invariants theory governed by these two new hierarchies. With this motivation, this paper will be partly aimed at constructing a kind of Hirota quadratic equation taking values in a matrix-valued differential algebra set. This kind of Hirota bilinear equations might be useful in Gromov-Witten theory and noncommutative symplectic geometry.

This paper is arranged as follows. In the next section we will recall the factorization problem and construct the logarithmic matrix operators using which we will define the extended flows of the multicomponent bigraded Toda hierarchy. In Sections 3, we will give the Lax equations of the extended multicomponent bigraded Toda hierarchy (EMTH), meanwhile the multicomponent Toda equations and the extended equations are introduced in this hierarchy. By Sato equations, Hirota bilinear equations of the EMTH are proved in Section 4. The tau function of the EMTH will be defined in Section 5 which leads to the formalism of the generalized matrix-valued vertex operators and Hirota quadratic equations in Section 6. In Section 7, the multi-Darboux transformation of the EMTH is constructed using determinant techniques. After this, to prove the integrability of the EMTH, the bi-Hamiltonian structure and tau symmetry of this hierarchy are given. The last section will be devoted to a short conclusion and discussion.

2 Factorization and logarithmic operators

In this section, we will denote G as a group which contains invertible elements of $N \times N$ complex matrices and denote its Lie algebra \mathfrak{g} as the associative algebra of $N \times N$ complex matrices $M_N(\mathbb{C})$. Now we will consider the linear space of functions $g : \mathbb{R} \rightarrow M_N(\mathbb{C})$ with the shift operator Λ acting on any functions $g(x)$ as $(\Lambda g)(x) := g(x + \epsilon)$. A Left multiplication by $X : \mathbb{R} \rightarrow M_N(\mathbb{C})$ is as $X\Lambda^j$, which acts on an arbitrary function $g(x)$ as $(X\Lambda^j)(g)(x) := X(x) \cdot g(x + j\epsilon)$. Also two operators $X(x)\Lambda^i$ and $Y(x)\Lambda^j$ have the product as $(X(x)\Lambda^i) \cdot (Y(x)\Lambda^j) := X(x)Y(x + i\epsilon)\Lambda^{i+j}$.

This Lie algebra as a linear space has the following important splitting

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \quad (2.1)$$

where

$$\mathfrak{g}_+ = \left\{ \sum_{j \geq 0} X_j(x) \Lambda^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}, \quad \mathfrak{g}_- = \left\{ \sum_{j < 0} X_j(x) \Lambda^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}.$$

The splitting (2.1) leads us to consider the following factorization of $g \in G$

$$g = g_-^{-1} \cdot g_+, \quad g_{\pm} \in G_{\pm} \quad (2.2)$$

where G_{\pm} have \mathfrak{g}_{\pm} as their Lie algebras. Here G_+ is the set of invertible linear operators of the form $\sum_{j \geq 0} g_j(x) \Lambda^j$; while G_- is the set of invertible linear operators in the form of $1 + \sum_{j < 0} g_j(x) \Lambda^j$.

Now we introduce the following free operators $W_0, \bar{W}_0 \in G$

$$W_0 := \sum_{k=1}^N E_{kk} e^{\sum_{j=0}^{\infty} t_{jk} \Lambda^j + s_j \frac{\Lambda^j}{j!} (\partial - c_j)}, \quad (2.3)$$

$$\bar{W}_0 := \sum_{k=1}^N E_{kk} e^{\sum_{j=0}^{\infty} \bar{t}_{jk} \Lambda^{-j} + s_j \frac{\Lambda^{-j}}{j!} (\partial - c_j)}, \quad (2.4)$$

where $t_{jk}, \bar{t}_{jk}, s_j \in \mathbb{C}$ will play the role of continuous times. We define the dressing operators W, \bar{W} as follows

$$W := S \cdot W_0, \quad \bar{W} := \bar{S} \cdot \bar{W}_0. \quad (2.5)$$

Given an element $g \in G$ and time series $t = (t_{jk}), \bar{t} = (\bar{t}_{jk}), s = (s_j); j, k \in \mathbb{N}, 1 \leq k \leq N$, one can consider the factorization problem in G [19]

$$W \cdot g = \bar{W}, \quad (2.6)$$

i.e. the factorization problem

$$S(t, \bar{t}, s) \cdot W_0 \cdot g = \bar{S}(\bar{t}, \bar{t}, s) \cdot \bar{W}_0, \quad S \in G_- \text{ and } \bar{S} \in G_+. \quad (2.7)$$

Observe that S, \bar{S} have expansions of the form

$$\begin{aligned} S &= \mathbb{I}_N + \omega_1(x) \Lambda^{-1} + \omega_2(x) \Lambda^{-2} + \cdots \in G_-, \\ \bar{S} &= \bar{\omega}_0(x) + \bar{\omega}_1(x) \Lambda + \bar{\omega}_2(x) \Lambda^2 + \cdots \in G_+. \end{aligned} \quad (2.8)$$

Also we define the symbols of S, \bar{S} as $\mathbb{S}, \bar{\mathbb{S}}$

$$\begin{aligned} \mathbb{S} &= \mathbb{I}_N + \omega_1(x) \lambda^{-1} + \omega_2(x) \lambda^{-2} + \cdots, \\ \bar{\mathbb{S}} &= \bar{\omega}_0(x) + \bar{\omega}_1(x) \lambda + \bar{\omega}_2(x) \lambda^2 + \cdots. \end{aligned} \quad (2.9)$$

Also the inverse operators S^{-1}, \bar{S}^{-1} of operators S, \bar{S} have expansions of the form

$$\begin{aligned} S^{-1} &= \mathbb{I}_N + \omega'_1(x) \Lambda^{-1} + \omega'_2(x) \Lambda^{-2} + \cdots \in G_-, \\ \bar{S}^{-1} &= \bar{\omega}'_0(x) + \bar{\omega}'_1(x) \Lambda + \bar{\omega}'_2(x) \Lambda^2 + \cdots \in G_+. \end{aligned} \quad (2.10)$$

Also we define the symbols of S^{-1}, \bar{S}^{-1} as $\mathbb{S}^{-1}, \bar{\mathbb{S}}^{-1}$

$$\begin{aligned} \mathbb{S}^{-1} &= \mathbb{I}_N + \omega'_1(x) \lambda^{-1} + \omega'_2(x) \lambda^{-2} + \cdots, \\ \bar{\mathbb{S}}^{-1} &= \bar{\omega}'_0(x) + \bar{\omega}'_1(x) \lambda + \bar{\omega}'_2(x) \lambda^2 + \cdots. \end{aligned} \quad (2.11)$$

The Lax operators $\mathcal{L}, C_{kk}, \bar{C}_{kk} \in \mathfrak{g}$ are defined by

$$\mathcal{L} := W \cdot \Lambda \cdot W^{-1} = \bar{W} \cdot \Lambda^{-1} \cdot \bar{W}^{-1}, \quad (2.12)$$

$$C_{kk} := W \cdot E_{kk} \cdot W^{-1}, \quad \bar{C}_{kk} := \bar{W} \cdot E_{kk} \cdot \bar{W}^{-1}, \quad (2.13)$$

and have the following expansions

$$\begin{aligned} \mathcal{L} &= \Lambda + u_1(x) + u_2(x)\Lambda^{-1}, \\ C_{kk} &= E_{kk} + C_{kk,1}(x)\Lambda^{-1} + C_{kk,2}(x)\Lambda^{-2} + \dots, \\ \bar{C}_{kk} &= \bar{C}_{kk,0}(x) + \bar{C}_{kk,1}(x)\Lambda + \bar{C}_{kk,2}(x)\Lambda^2 + \dots. \end{aligned} \quad (2.14)$$

In fact the Lax operators $\mathcal{L}, C_{kk}, \bar{C}_{kk} \in \mathfrak{g}$ can also be equivalently defined by

$$\mathcal{L} := S \cdot \Lambda \cdot S^{-1} = \bar{S} \cdot \Lambda^{-1} \cdot \bar{S}^{-1}, \quad (2.15)$$

$$C_{kk} := S \cdot E_{kk} \cdot S^{-1}, \quad \bar{C}_{kk} := \bar{S} \cdot E_{kk} \cdot \bar{S}^{-1}. \quad (2.16)$$

These definitions are continuous interpolated versions of the multi-component Toda hierarchy, i.e. a continuous spatial parameter x is brought into this hierarchy. Under this meaning, the continuous flow $\frac{\partial}{\partial x}$ is missing. To make these flows complete, we define the following logarithmic matrices

$$\log_+ \mathcal{L} = (S \cdot \epsilon \partial \cdot S^{-1}) = \epsilon \partial + \sum_{k < 0} W_k(x) \Lambda^k, \quad (2.17)$$

$$\log_- \mathcal{L} = -(\bar{S} \cdot \epsilon \partial \cdot \bar{S}^{-1}) = -\epsilon \partial + \sum_{k \geq 0} W_k(x) \Lambda^k, \quad (2.18)$$

where ∂ is the derivative about spatial variable x .

Combining these above logarithmic operators together can help us in deriving the following important logarithmic matrix

$$\log \mathcal{L} := \frac{1}{2} \log_+ \mathcal{L} + \frac{1}{2} \log_- \mathcal{L} = \frac{1}{2} (S \cdot \epsilon \partial \cdot S^{-1} - \bar{S} \cdot \epsilon \partial \cdot \bar{S}^{-1}), \quad (2.19)$$

which will generate a series of flow equations which contain the spatial flow in Lax equations.

3 Lax equations of EMTH

In this section we will use the factorization problem (2.6) to derive Lax equations. Let us first introduce some convenient notations.

Definition 1. *The matrix operators $C_{kk}, \bar{C}_{kk}, B_{jk}, \bar{B}_{jk}, D_j$ are defined as follows*

$$\begin{aligned} C_{kk} &:= W E_{kk} W^{-1}, \quad \bar{C}_{kk} := \bar{W} E_{kk} \bar{W}^{-1}, \\ B_{jk} &:= W E_{kk} \Lambda^j W^{-1}, \quad \bar{B}_{jk} := \bar{W} E_{kk} \Lambda^{-j} \bar{W}^{-1}, \\ D_j &:= \frac{2\mathcal{L}^j}{j!} (\log \mathcal{L} - c_j), \quad c_0 = 0; \quad c_j = \sum_{i=1}^j \frac{1}{i}, \quad j \geq 1. \end{aligned} \quad (3.1)$$

Now we give the definition of the extended multicomponent Toda hierarchy (EMTH).

Definition 2. *The extended multicomponent Toda hierarchy is a hierarchy in which the dressing operators S, \bar{S} satisfy following Sato equations*

$$\epsilon \partial_{t_{jk}} S = -(B_{jk})_- S, \quad \epsilon \partial_{t_{jk}} \bar{S} = (B_{jk})_+ \cdot \bar{S}, \quad (3.2)$$

$$\epsilon \partial_{\bar{t}_{jk}} S = -(\bar{B}_{jk})_- \cdot S, \quad \epsilon \partial_{\bar{t}_{jk}} \bar{S} = (\bar{B}_{jk})_+ \cdot \bar{S}, \quad (3.3)$$

$$\epsilon \partial_{s_j} S = -(D_j)_- \cdot S, \quad \epsilon \partial_{s_j} \bar{S} = (D_j)_+ \cdot \bar{S}. \quad (3.4)$$

Then one can easily get the following proposition about W, \bar{W} .

Proposition 1. *The wave operators W, \bar{W} satisfy following Sato equations*

$$\epsilon \partial_{t_{jk}} W = (B_{jk})_+ \cdot W, \quad \epsilon \partial_{t_{jk}} \bar{W} = (B_{jk})_+ \cdot \bar{W}, \quad (3.5)$$

$$\epsilon \partial_{\bar{t}_{jk}} W = -(\bar{B}_{jk})_- \cdot W, \quad \epsilon \partial_{\bar{t}_{jk}} \bar{W} = -(\bar{B}_{jk})_- \cdot \bar{W}, \quad (3.6)$$

$$\epsilon \partial_{s_j} W = \left(\frac{\mathcal{L}^j}{j!} (\log_+ \mathcal{L} - c_j) - (D_j)_- \right) \cdot W, \quad \epsilon \partial_{s_j} \bar{W} = \left(-\frac{\mathcal{L}^j}{j!} (\log_- \mathcal{L} - c_j) + (D_j)_+ \right) \cdot \bar{W}. \quad (3.7)$$

From the previous proposition we can derive the following Lax equations for the Lax operators.

Proposition 2. *The Lax equations of the EMTH are as follows*

$$\epsilon \partial_{t_{jk}} \mathcal{L} = [(B_{jk})_+, \mathcal{L}], \quad \epsilon \partial_{t_{jk}} C_{ss} = [(B_{jk})_+, C_{ss}], \quad \epsilon \partial_{t_{jk}} \bar{C}_{ss} = [(B_{jk})_+, \bar{C}_{ss}], \quad (3.8)$$

$$\epsilon \partial_{\bar{t}_{jk}} \mathcal{L} = [(\bar{B}_{jk})_+, \mathcal{L}], \quad \epsilon \partial_{\bar{t}_{jk}} C_{ss} = [(\bar{B}_{jk})_+, C_{ss}], \quad \epsilon \partial_{\bar{t}_{jk}} \bar{C}_{ss} = [(\bar{B}_{jk})_+, \bar{C}_{ss}], \quad (3.9)$$

$$\epsilon \partial_{s_j} \mathcal{L} = [(D_j)_+, \mathcal{L}], \quad \epsilon \partial_{s_j} C_{ss} = [(D_j)_+, C_{ss}], \quad \epsilon \partial_{s_j} \bar{C}_{ss} = [(D_j)_+, \bar{C}_{ss}], \quad (3.10)$$

$$\epsilon \partial_{t_{jk}} \log \mathcal{L} = [(B_{jk})_+, \log \mathcal{L}], \quad \epsilon \partial_{\bar{t}_{jk}} \log \mathcal{L} = [-(\bar{B}_{jk})_-, \log \mathcal{L}], \quad (3.11)$$

$$\epsilon (\log \mathcal{L})_{s_j} = [-(D_j)_-, \frac{1}{2} \log_+ \mathcal{L}] + [(D_j)_+, \frac{1}{2} \log_- \mathcal{L}]. \quad (3.12)$$

Proof. By eq.(3.5) and eq.(3.6), one can get eq.(3.11) using dressing structures.

Similarly, the other Lax equations of EMTH can be derived using the double dressing structures and Sato equations eq.(3.2)-eq.(3.4). \square

To see this kind of hierarchy more clearly, the $\partial_{t_{1k}}$ flow equations will be given in the next subsection.

3.1 The multicomponent Toda equations

As a consequence of the factorization problem (2.6) and Sato equations, after taking into account that $S \in G_-$ and $\bar{S} \in G_+$ and using the notation $e^\phi := \bar{\omega}_0$ in \bar{S} , B_{1k} has following form

$$B_{1k} = E_{kk} \Lambda + U_k + \bar{U}_k \Lambda^{-1}, \quad 1 \leq k \leq N, \quad (3.13)$$

and we have the alternative expressions

$$\begin{aligned} U_k &:= \omega_1(x) E_{kk} - E_{kk} \omega_1(x + \epsilon) = \epsilon \partial_{t_{1k}} (e^{\phi(x)}) \cdot e^{-\phi(x)}, \\ \bar{U}_k &= e^{\phi(x)} E_{kk} e^{-\phi(x-\epsilon)} = -\epsilon \partial_{t_{1k}} \omega_1(x). \end{aligned} \quad (3.14)$$

From Sato equations we deduce the following set of nonlinear partial differential-difference equations

$$\begin{cases} \omega_1(x) E_{kk} - E_{kk} \omega_1(x + \epsilon) = \epsilon \partial_{t_{1k}} (e^{\phi(x)}) \cdot e^{-\phi(x)}, \\ \epsilon \partial_{t_{1k}} \omega_1(x) = -e^{\phi(x)} E_{kk} e^{-\phi(x-\epsilon)}. \end{cases} \quad (3.15)$$

These equations constitute what we call the multicomponent Toda equations. Observe that if we cross the two equations in (3.15), then we get

$$\epsilon \partial_{t_{1k}} (\epsilon \partial_{t_{1k}} (e^{\phi(x)}) \cdot e^{-\phi(x)}) = E_{kk} e^{\phi(x+\epsilon)} E_{kk} e^{-\phi(x)} - e^{\phi(x)} E_{kk} e^{-\phi(x-\epsilon)} E_{kk},$$

which is the matrix extension of the following Toda equation (the case when $N = 1$)

$$\epsilon^2 \partial_{t_{11}} \partial_{t_{11}} (\phi(x)) = e^{\phi(x+\epsilon)-\phi(x)} - e^{\phi(x)-\phi(x-\epsilon)}.$$

Besides above multicomponent Toda equations, the logarithmic flows the EMTH also contains some extended flow equations in the next subsection.

3.2 The extended equations

Here we consider what the extended flow equations are like. Here we take the simplest case, i.e. the s_0 flow for $\mathcal{L} = \Lambda + u_0 + u_1 \Lambda^{-1}$,

$$\epsilon \partial_{s_0} \mathcal{L} = [(S \epsilon \partial_x S^{-1})_+, \mathcal{L}] \quad (3.16)$$

$$= [\epsilon \partial_x S S^{-1}, \mathcal{L}] \quad (3.17)$$

$$= \epsilon \mathcal{L}_x, \quad (3.18)$$

which leads to the following specific equation

$$\partial_{s_0} u_0 = u_{0x}, \quad \partial_{s_0} u_1 = u_{1x}. \quad (3.19)$$

The above flow equation tells us that ∂_{s_0} is just the ∂_x flow.

To give a linear description of the EMTH, we introduce matrix wave functions $\psi, \bar{\psi}$ in the following part. The matrix wave functions of the multi-component Toda hierarchy are defined by

$$\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \bar{\chi}, \quad (3.20)$$

where

$$\chi(z) := z^{\frac{x}{\epsilon}} \mathbb{I}_N, \quad \bar{\chi}(z) := z^{-\frac{x}{\epsilon}} \mathbb{I}_N. \quad (3.21)$$

Note that $\Lambda \chi = z \chi$ and the following asymptotic expansions are consequences of (2.8)

$$\begin{aligned} \psi &= z^{\frac{x}{\epsilon}} (\mathbb{I}_N + \omega_1(x) z^{-1} + \cdots) \psi_0(z), \quad \psi_0 := \sum_{k=1}^N E_{kk} e^{\sum_{j=1}^{\infty} t_{jk} z^j + s_j z^j \log z}, \quad z \rightarrow \infty, \\ \bar{\psi} &= z^{-\frac{x}{\epsilon}} (\bar{\omega}_0(x) + \bar{\omega}_1(x) z + \cdots) \bar{\psi}_0(z), \quad \bar{\psi}_0 := \sum_{k=1}^N E_{kk} e^{\sum_{j=1}^{\infty} \bar{t}_{jk} z^{-j} + s_j z^{-j} \log z}, \quad z \rightarrow 0. \end{aligned} \quad (3.22)$$

We can further get linear equations in the following proposition.

Proposition 3. *The matrix wave functions $\psi, \bar{\psi}$ are subject to following Sato equations*

$$\mathcal{L}\psi = z\psi, \quad \mathcal{L}\bar{\psi} = z\bar{\psi}, \quad (3.23)$$

$$\epsilon\partial_{jk}\psi = (B_{jk})_+ \cdot \psi, \quad \epsilon\partial_{jk}\bar{\psi} = (B_{jk})_+ \cdot \bar{\psi}, \quad (3.24)$$

$$\epsilon\partial_{\bar{t}_{jk}}\psi = -(\bar{B}_{jk})_- \cdot \psi, \quad \epsilon\partial_{\bar{t}_{jk}}\bar{\psi} = -(\bar{B}_{jk})_- \cdot \bar{\psi}, \quad (3.25)$$

$$\epsilon\partial_{s_j}\psi = (C_{ss}\mathcal{L}^j \log_+ \mathcal{L} - (D_j)_-) \cdot \psi, \quad \epsilon\partial_{s_j}\bar{\psi} = (-\bar{C}_{ss}\mathcal{L}^j \log_- \mathcal{L} + (D_j)_+) \cdot \bar{\psi}. \quad (3.26)$$

4 Hirota bilinear equations

Basing on the above section, the Hirota bilinear equations which are equivalent to Lax equations of the EMTH can be derived in following proposition.

Proposition 4. *W and \bar{W} are matrix wave operators of the multicomponent Toda hierarchy if and only the following Hirota bilinear equations hold true*

$$W\Lambda^r W^{-1} = \bar{W}\Lambda^{-r}\bar{W}^{-1}, \quad r \in \mathbb{N}. \quad (4.1)$$

Proof. \Rightarrow Firstly we will set

$$\alpha = (\alpha_{0,1}, \alpha_{1,1}, \alpha_{2,1}, \dots; \alpha_{0,2}, \alpha_{1,2}, \alpha_{2,2}, \dots; \dots; \alpha_{0,N}, \alpha_{1,N}, \alpha_{2,N}, \dots;), \quad (4.2)$$

$$\bar{\alpha} = (\bar{\alpha}_{0,1}, \bar{\alpha}_{1,1}, \bar{\alpha}_{2,1}, \dots; \bar{\alpha}_{0,2}, \bar{\alpha}_{1,2}, \bar{\alpha}_{2,2}, \dots; \dots; \bar{\alpha}_{0,N}, \bar{\alpha}_{1,N}, \bar{\alpha}_{2,N}, \dots;), \quad (4.3)$$

$$\beta = (\beta_1, \beta_2, \dots), \quad (4.4)$$

be a multi index and

$$\partial^\alpha := \partial_{t_{0,1}}^{\alpha_{0,1}} \partial_{t_{1,1}}^{\alpha_{1,1}} \partial_{t_{2,1}}^{\alpha_{2,1}} \dots; \partial_{t_{0,2}}^{\alpha_{0,2}} \partial_{t_{1,2}}^{\alpha_{1,2}} \partial_{t_{2,2}}^{\alpha_{2,2}} \dots; \dots; \partial_{t_{0,N}}^{\alpha_{0,N}} \partial_{t_{1,N}}^{\alpha_{1,N}} \partial_{t_{2,N}}^{\alpha_{2,N}} \dots, \quad (4.5)$$

$$\partial^{\bar{\alpha}} := \partial_{\bar{t}_{0,1}}^{\bar{\alpha}_{0,1}} \partial_{\bar{t}_{1,1}}^{\bar{\alpha}_{1,1}} \partial_{\bar{t}_{2,1}}^{\bar{\alpha}_{2,1}} \dots; \partial_{\bar{t}_{0,2}}^{\bar{\alpha}_{0,2}} \partial_{\bar{t}_{1,2}}^{\bar{\alpha}_{1,2}} \partial_{\bar{t}_{2,2}}^{\bar{\alpha}_{2,2}} \dots; \dots; \partial_{\bar{t}_{0,N}}^{\bar{\alpha}_{0,N}} \partial_{\bar{t}_{1,N}}^{\bar{\alpha}_{1,N}} \partial_{\bar{t}_{2,N}}^{\bar{\alpha}_{2,N}} \dots, \quad (4.6)$$

$$\partial^\beta := \partial_{s_1}^{\beta_1} \partial_{s_2}^{\beta_2} \dots. \quad (4.7)$$

Then we suppose $\partial^\theta = \partial^\alpha \partial^{\bar{\alpha}} \partial^\beta$ (we stress that ∂_{s_0} is not involved). We shall prove the left statement leads to

$$W(x, t, \bar{t}, \Lambda) \Lambda^r W^{-1}(x, t', \Lambda) = \bar{W}(x, t, \bar{t}, \Lambda) \Lambda^{-r} \bar{W}^{-1}(x, t', \bar{t}', \Lambda) \quad (4.8)$$

for all integers $r \geq 0$. Using the same method used in [7, 9], by induction on θ , we shall prove that

$$W(x, t, \bar{t}, \Lambda) \Lambda^r (\partial^\theta W^{-1}(x, t, \bar{t}, \Lambda)) = \bar{W}(x, t, \bar{t}, \Lambda) \Lambda^{-r} (\partial^\theta \bar{W}^{-1}(x, t, \bar{t}, \Lambda)). \quad (4.9)$$

When $\theta = 0$, it is obviously true according to the definition of matrix wave operators.

Here firstly we suppose eq.(4.9) is true in the case of $\theta \neq 0$. Note that

$$\epsilon\partial_{p_{jk}} W := \begin{cases} [(\partial_{t_{jk}} S) S^{-1} + S E_{kk} \Lambda^j S^{-1}] W, & p_{jk} = t_{jk}, \\ (\partial_{\bar{t}_{jk}} S) S^{-1} W, & p_{jk} = \bar{t}_{jk}, \\ [(\partial_{s_j} S) S^{-1} + S \Lambda^j \partial_x S^{-1}] W, & p_{jk} = s_j, \end{cases}$$

and

$$\epsilon \partial_{p_{jk}} \bar{W} := \begin{cases} (\partial_{t_{jk}} \bar{S}) \bar{S}^{-1} \bar{W}, & p_{jk} = t_{jk}, \\ [(\partial_{\bar{t}_{jk}} \bar{S}) \bar{S}^{-1} + \bar{S} E_{kk} \Lambda^{-j} \bar{S}^{-1}] \bar{W}, & p_{jk} = \bar{t}_{jk}, \\ [(\partial_{s_j} \bar{S}) \bar{S}^{-1} + \bar{S} \Lambda^{-j} \partial_x \bar{S}^{-1}] \bar{W}, & p_{jk} = s_j, \end{cases}$$

which further lead to

$$\epsilon \partial_{p_{jk}} W := \begin{cases} (B_{jk})_+ W, & p_{jk} = t_{jk}, \\ -(\bar{B}_{jk})_- W, & p_{jk} = \bar{t}_{jk}, \\ [-(D_j)_- + \frac{\mathcal{L}^j}{j!} (\log_+ \mathcal{L} - c_j)] W, & p_{jk} = s_j, \end{cases}$$

and

$$\epsilon \partial_{p_{jk}} \bar{W} := \begin{cases} (B_{jk})_+ \bar{W}, & p_{jk} = t_{jk}, \\ -(\bar{B}_{jk})_- \bar{W}, & p_{jk} = \bar{t}_{jk}, \\ [(D_j)_+ - \frac{\mathcal{L}^j}{j!} (\log_- \mathcal{L} - c_j)] \bar{W}, & p_{jk} = s_j. \end{cases}$$

This further implies

$$(\partial_{p_{jk}} W) \Lambda^r (\partial^\theta W^{-1}) = (\partial_{p_{jk}} \bar{W}) \Lambda^{-r} (\partial^\theta \bar{W}^{-1})$$

by considering (4.9) and furthermore we get

$$W \Lambda^r (\partial_{p_{jk}} \partial^\theta W^{-1}) = \bar{W} \Lambda^{-r} (\partial_{p_{jk}} \partial^\theta \bar{W}^{-1}).$$

Thus if we increase the power of $\partial_{p_{jk}}$ by 1, the eq.(4.9) still holds true. The induction is completed. After doing Taylor expansion on both sides of eq.(4.8) about $t = t', \bar{t} = \bar{t}', s = s'$, one can finish the proof of eq.(4.8).

\Leftarrow Vice versa, by separating the negative and the positive projection part about powers of shift operator Λ in the equation (4.1), we can prove S, \bar{S} are a pair of matrix wave operators. \square

To give a description in terms of matrix wave functions, following symbolic definitions are needed. If the wave operator series have forms

$$W(x, t, \bar{t}, s, \Lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t, \bar{t}, s, \partial_x) \Lambda^i \text{ and } \bar{W}(x, t, \bar{t}, s, \Lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t, \bar{t}, s, \partial_x) \Lambda^i,$$

$$W^{-1}(x, t, \bar{t}, s, \Lambda) = \sum_{i \in \mathbb{Z}} \Lambda^i a'_i(x, t, \bar{t}, s, \partial_x) \text{ and } \bar{W}^{-1}(x, t, \bar{t}, s, \Lambda) = \sum_{j \in \mathbb{Z}} \Lambda^j b'_j(x, t, \bar{t}, s, \partial_x),$$

then we denote their corresponding left symbols $\mathcal{W}, \bar{\mathcal{W}}$ and right symbols $\mathcal{W}^{-1}, \bar{\mathcal{W}}^{-1}$ as following

$$\begin{aligned} \mathcal{W}(x, t, \bar{t}, s, \lambda) &= \sum_{i \in \mathbb{Z}} a_i(x, t, \bar{t}, s, \partial_x) \lambda^i, & \mathcal{W}^{-1}(x, t, \bar{t}, s, \lambda) &= \sum_{i \in \mathbb{Z}} a'_i(x, t, \bar{t}, s, \partial_x) \lambda^i, \\ \bar{\mathcal{W}}(x, t, \bar{t}, s, \lambda) &= \sum_{i \in \mathbb{Z}} b_i(x, t, \bar{t}, s, \partial_x) \lambda^i, & \bar{\mathcal{W}}^{-1}(x, t, \bar{t}, s, \lambda) &= \sum_{j \in \mathbb{Z}} b'_j(x, t, \bar{t}, s, \partial_x) \lambda^j. \end{aligned}$$

With above preparation, it is time to give another form of Hirota bilinear equations (see the following proposition) after defining residue as $\text{Res}_\lambda \sum_{n \in \mathbb{Z}} \alpha_n \lambda^n = \alpha_{-1}$ using the similar proof as [3, 7, 9].

Proposition 5. *Let $s_0 = s'_0$, S and \bar{S} are matrix-valued wave operators of the multicomponent Toda hierarchy if and only if for all $m \in \mathbb{Z}$, $r \in \mathbb{N}$, the following Hirota bilinear identity holds*

$$\begin{aligned} \text{Res}_\lambda \left\{ \lambda^{r+m-1} \mathcal{W}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \mathcal{W}^{-1}(x - m\epsilon, t', \bar{t}', s, \epsilon \partial_x, \lambda) \right\} = \\ \text{Res}_\lambda \left\{ \lambda^{-r+m-1} \bar{\mathcal{W}}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \bar{\mathcal{W}}^{-1}(x - m\epsilon, t', \bar{t}', s', \epsilon \partial_x, \lambda) \right\}. \end{aligned} \quad (4.10)$$

Proof. Let $m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $s_0 = s'_0$, then one can compare the coefficients in front of Λ^{-m} on both sides of eq.(4.8) and find:

$$\sum_{i+j=-m-r} a_i(x, t, \bar{t}, s, \partial_x) a'_j(x - m\epsilon, t', \bar{t}', s', \partial_x) = \sum_{i+j=-m+r} b_i(x, t, \bar{t}, s, \partial_x) b'_j(x - m\epsilon, t', \bar{t}', s', \partial_x).$$

This equality can be also rewritten as eq.(4.10). □

To give Hirota quadratic functions in terms of tau functions, we need to define tau functions and prove the existence of tau functions of the EMTH.

5 Tau-functions of EMTH

Firstly, we need to introduce the following sequences:

$$t - [\lambda] := \left(t_{jk} - \frac{\epsilon \lambda^j}{j}, 0 \leq j \leq \infty, 1 \leq k \leq N \right), \quad (5.1)$$

$$\bar{t} - [\lambda] := \left(\bar{t}_{jk} - \frac{\epsilon \lambda^j}{j}, 0 \leq j \leq \infty, 1 \leq k \leq N \right). \quad (5.2)$$

Referring to the Proposition 35 in [21], the following tau function can be defined. The matrix functions $\tau_M, \bar{\tau}_M$ depending only on the dynamical variables t, \bar{t} and ϵ are called the **Matrix tau-functions of the EMTH** if they are related to the symbols of the

matrix wave operators as following,

$$(\mathbb{S})_{kk} : = \frac{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t_{js} - \delta_{s,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.3)$$

$$(\mathbb{S})_{sk} : = \lambda^{-1} \frac{\tau_{sk}(s_{0,k} + x - \frac{\epsilon}{2}, t_{jr} - \delta_{r,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad s \neq k, \quad (5.4)$$

$$(\mathbb{S}^{-1})_{kk} : = \frac{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t_{js} + \delta_{s,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.5)$$

$$(\mathbb{S}^{-1})_{sk} : = \lambda \frac{\tau_{sk}(s_{0,k} + x + \frac{\epsilon}{2}, t_{jr} + \delta_{r,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad s \neq k, \quad (5.6)$$

$$(\bar{\mathbb{S}})_{kk} : = \frac{\tau_{kk}(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}_{js} + \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.7)$$

$$(\bar{\mathbb{S}})_{sk} : = \frac{\bar{\tau}_{sk}(s_{0,k} + x + \frac{3\epsilon}{2}, t, \bar{t}_{js} + \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad s \neq k, \quad (5.8)$$

$$(\bar{\mathbb{S}}^{-1})_{kk} : = \frac{\tau_{kk}(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}_{js} - \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.9)$$

$$(\bar{\mathbb{S}}^{-1})_{sk} : = \frac{\bar{\tau}_{sk}(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}_{js} - \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{3\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad s \neq k. \quad (5.10)$$

For convenience, we denote two matrices τ_M and $\bar{\tau}_M$ as

$$(\tau_M)_{ij} = \begin{cases} \tau_{ii} = \tau, & i = j \\ \tau_{ij}, & i \neq j, \end{cases} \quad (\bar{\tau}_M)_{ij} = \begin{cases} \bar{\tau}_{ii}, & i = j \\ \bar{\tau}_{ij}, & i \neq j. \end{cases} \quad (5.11)$$

Then we can rewrite the definition of tau functions as

$$\mathbb{S} : = \frac{\tau_M(s_{0,k} + x - \frac{\epsilon}{2}, t_{js} - \delta_{s,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.12)$$

$$\mathbb{S}^{-1} : = \frac{\tau_M(s_{0,k} + x + \frac{\epsilon}{2}, t_{js} + \delta_{s,k} \frac{\epsilon}{j\lambda^j}, \bar{t}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.13)$$

$$\bar{\mathbb{S}} : = \frac{\bar{\tau}_M(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}_{js} + \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}, \quad (5.14)$$

$$\bar{\mathbb{S}}^{-1} : = \frac{\bar{\tau}_M(s_{0,k} + x - \frac{\epsilon}{2}, t, \bar{t}_{js} - \delta_{s,k} \frac{\epsilon\lambda^j}{j}, s; \epsilon)}{\tau(s_{0,k} + x + \frac{\epsilon}{2}, t, \bar{t}, s; \epsilon)}. \quad (5.15)$$

To give the existence of tau function, we need the following two lemmas.

Lemma 1. *The following identities about $\mathbb{S}(x, t, \lambda), \mathbb{S}^{-1}(x, t, \lambda), \bar{\mathbb{S}}(x, t, \lambda), \bar{\mathbb{S}}^{-1}(x, t, \lambda)$ hold*

$$\mathbb{S}(x, \mathbf{t}, \lambda_1)_{kk} \mathbb{S}^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k, \lambda_1)_{kk} \quad (5.16)$$

$$= \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_2)_{kk} \bar{\mathbb{S}}^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k, \lambda_2)_{kk} + \sum_{s \neq k} \tilde{\omega}_0(x, \mathbf{t})_{ks} \tilde{\omega}_0^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k)_{sk},$$

$$\mathbb{S}(x, \mathbf{t}, \lambda)_{kk} \mathbb{S}^{-1}(x - \epsilon, t - [\lambda^{-1}]_k, \bar{t}, \lambda)_{kk} = 1, \quad (5.17)$$

$$\begin{aligned} & \mathbb{S}(x, \mathbf{t}, \lambda_1)_{kk} \mathbb{S}^{-1}(x - \epsilon, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \bar{t}, \lambda_1)_{kk} \\ &= \mathbb{S}(x, \mathbf{t}, \lambda_2)_{kk} \mathbb{S}^{-1}(x - \epsilon, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \lambda_2)_{kk}, \end{aligned} \quad (5.18)$$

$$\bar{\mathbb{S}}(x, \mathbf{t}, \lambda_1)_{kk} \bar{\mathbb{S}}^{-1}(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k, \lambda_1)_{kk} \quad (5.19)$$

$$\begin{aligned} &= \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_2)_{kk} \bar{\mathbb{S}}^{-1}(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k, \lambda_2)_{kk} + (\lambda_2 - \lambda_1) \tilde{\omega}'_1(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k)_{kk}, \\ & \bar{\mathbb{S}}(x, \mathbf{t}, \lambda)_{kk} \bar{\mathbb{S}}^{-1}(x, \bar{t} + [\lambda]_k, \lambda)_{kk} + \sum_{s \neq k} \tilde{\omega}_0(x, \mathbf{t})_{ks} \tilde{\omega}_0^{-1}(x, \bar{t} + [\lambda]_k)_{sk} = 1. \end{aligned} \quad (5.20)$$

In the following part, we sometimes denote \mathbf{t} as (t, \bar{t}) for short.

Lemma 2. *The following identities about $\mathbb{S}(x, t, \lambda), \mathbb{S}^{-1}(x, t, \lambda), \bar{\mathbb{S}}(x, t, \lambda), \bar{\mathbb{S}}^{-1}(x, t, \lambda)$ hold*

$$\mathbb{S}(x, t, \lambda_1)_{ik} \mathbb{S}^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k, \lambda_1)_{kj} + \delta_{ij} - \delta_{ik} \delta_{kj} \quad (5.21)$$

$$= \bar{\mathbb{S}}(x, t, \lambda_2)_{ik} \bar{\mathbb{S}}^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k, \lambda_2)_{kj} + \sum_{s \neq k} \tilde{\omega}_0(x, \mathbf{t})_{is} \tilde{\omega}_0^{-1}(x, t - [\lambda_1^{-1}]_k, \bar{t} + [\lambda_2]_k)_{sj},$$

$$\begin{aligned} & \omega_1(x, \mathbf{t})_{ij} + \delta_{ik} \omega_1(x, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \bar{t}, \lambda_1)_{kj} + \mathbb{S}(x, \mathbf{t}, \lambda_1)_{ik} \mathbb{S}^{-1}(x - \epsilon, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \bar{t}, \lambda_1)_{kj} \\ &= \delta_{jk} \omega_1(x, \mathbf{t})_{ik} + \omega_1(x, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \bar{t}, \lambda_1)_{kj} + \mathbb{S}(x, \mathbf{t}, \lambda_2)_{ik} \mathbb{S}^{-1}(x - \epsilon, t - [\lambda_1^{-1}]_k - [\lambda_2^{-1}]_k, \lambda_2)_{kj}, \\ & \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_1)_{rk} \bar{\mathbb{S}}^{-1}(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k, \lambda_1)_{kk} - \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_2)_{rk} \bar{\mathbb{S}}^{-1}(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k, \lambda_2)_{kk} \\ &= (\lambda_2 - \lambda_1) \left[\sum_{s \neq k} \tilde{\omega}_1(x, t)_{rs} \tilde{\omega}_0^{-1}(x + \epsilon, \bar{t} + [\lambda_1]_k + [\lambda_2]_k)_{sk} - \sum_{s \neq k} \tilde{\omega}_0(x, t)_{rs} \tilde{\omega}'_1(x, \bar{t} + [\lambda_1]_k + [\lambda_2]_k)_{sk} \right], \end{aligned}$$

$$\bar{\mathbb{S}}(x, \mathbf{t}, \lambda)_{ik} \bar{\mathbb{S}}^{-1}(x, \bar{t} + [\lambda]_k, \lambda)_{kj} + \sum_{s \neq k} \tilde{\omega}_0(x, \mathbf{t})_{is} \tilde{\omega}_0^{-1}(x, \bar{t} + [\lambda]_k)_{sj} = \delta_{ij}, \quad (5.22)$$

$$\mathbb{S}(x, \mathbf{t}, \lambda_1)_{ss} \mathbb{S}^{-1}(x, \mathbf{t} - [\lambda_1^{-1}]_s, \lambda_1)_{ss} = \sum_k \tilde{\omega}_0(x, \mathbf{t})_{sk} \tilde{\omega}_0^{-1}(x, \mathbf{t} - [\lambda_1^{-1}]_k, \bar{t})_{ks}. \quad (5.23)$$

Proof. Let \mathbf{t} and \mathbf{t}' be two sequences of time variables such that $s_n = s'_n$, $n \geq 0$. The identities (5.21) – (5.22) are consequence of the following one:

$$\begin{aligned} & S(x, \mathbf{t}, \Lambda) \sum_{k=1}^N E_{kk} \exp\left(\sum_{j=1}^{\infty} (t_{jk} - t'_{jk}) \Lambda^j\right) S^{-1}(x, \mathbf{t}', \Lambda) \\ &= \bar{S}(x, \mathbf{t}, \Lambda) \sum_{k=1}^N E_{kk} \exp\left(-\sum_{j=1}^{\infty} (\bar{t}_{jk} - \bar{t}'_{jk}) \Lambda^{-j}\right) \bar{S}^{-1}(x, \mathbf{t}', \Lambda). \end{aligned} \quad (5.24)$$

The proof of (5.24) is completely analogous to the argument in the implication from left statement to the right result in Proposition 4 and it will be omitted. To prove eq.(5.21), in eq.(5.24) we put $t'_{jk} = t_{jk} - [\lambda_1^{-1}]_k + [\lambda_2]_k$. The exponential factor turns into

$$\exp\left(\sum_{j=0}^{\infty} \Lambda^j \lambda^{-j}\right) = (1 - \lambda^{-1} \Lambda)^{-1} = \sum_{s \geq 0} (\lambda^{-1} \Lambda)^s,$$

$$\exp \left(\sum_{j=0}^{\infty} \lambda^j \Lambda^{-j} \right) = (1 - \lambda \Lambda^{-1})^{-1} = \sum_{s \geq 0} (\lambda \Lambda^{-1})^s.$$

The other identities can be proved similarly as [7]. \square

The above two lemmas can be rewritten into the following single lemma.

Lemma 3. *The following equations hold*

$$\sum_{k=1}^N \mathbb{S}(x, \mathbf{t}, \lambda_{1k})_{ik} \mathbb{S}^{-1}(x + \epsilon, t, \bar{t} + [\lambda_{2k}]_k, \lambda_{1k})_{kj} = \sum_{k=1}^N \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_{2k})_{ik} \bar{\mathbb{S}}^{-1}(x, t - [\lambda_{1k}^{-1}]_k, \bar{t}, \lambda_{2k})_{kj}, \quad (5.25)$$

$$\sum_{k=1}^N \mathbb{S}(x, \mathbf{t}, \lambda_{1k})_{ik} \mathbb{S}^{-1}(x, t - [\lambda_{2k}^{-1}]_k, \bar{t}, \lambda_{1k})_{kj} = \sum_{k=1}^N \mathbb{S}(x, \mathbf{t}, \lambda_{2k})_{ik} \mathbb{S}^{-1}(x, t - [\lambda_{1k}^{-1}]_k, \bar{t}, \lambda_{2k})_{kj}, \quad (5.26)$$

$$\sum_{k=1}^N \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_{1k})_{ik} \bar{\mathbb{S}}^{-1}(x + \epsilon, t, \bar{t} + [\lambda_{2k}]_k, \lambda_{1k})_{kj} = \sum_{k=1}^N \bar{\mathbb{S}}(x, \mathbf{t}, \lambda_{2k})_{ik} \bar{\mathbb{S}}^{-1}(x + \epsilon, t, \bar{t} + [\lambda_{1k}]_k, \lambda_{2k})_{kj}. \quad (5.27)$$

Using these lemmas above, one can prove the existence of tau function in the following proposition.

Proposition 6. *Given a pair of wave operators \mathbb{S} and $\bar{\mathbb{S}}$ of the METH there exists corresponding tau-functions $\tau_{sk}, \bar{\tau}_{sk}$, which is unique up to multiplication by a non-vanishing function independent of t_{sk}, \bar{t}_{sk} .*

Proof. The proof is quite complicated but standard which will not be mentioned here. One can check the references [7, 9, 21]. \square

With the above preparation, we will give the Hirota bilinear equations in terms of tau functions in the next section with the help of generalized vertex operators.

6 Generalized matrix vertex operators and Hirota quadratic equations

In this section we continue to discuss on the fundamental properties of the tau function of the EMTH, i.e., the Hirota quadratic equations of the EMTH. So we introduce the following vertex operators

$$\begin{aligned} \Gamma^{\pm a} : &= \exp \left(\pm \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \left(\sum_{j=0}^{\infty} t_{jk} \lambda^j + s_j \lambda^j \log \lambda \right) \right) \times \exp \left(\mp \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda^{-1}]_{\partial} \right), \\ \Gamma^{\pm b} : &= \exp \left(\pm \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \left(\sum_{j=0}^{\infty} \bar{t}_{jk} \lambda^{-j} - s_j \lambda^{-j} \log \lambda \right) \right) \times \exp \left(\mp \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda]_{\bar{\partial}} \right). \end{aligned}$$

where

$$[\lambda]_{\partial} := \epsilon \sum_{k=1}^N \sum_{j=0}^{\infty} \frac{\lambda^j}{j} \partial_{t_{jk}}, \quad [\lambda]_{\bar{\partial}} := \epsilon \sum_{k=1}^N \sum_{j=0}^{\infty} \frac{\lambda^j}{j} \partial_{\bar{t}_{jk}}.$$

Because of the logarithm $\log \lambda$, the vertex operators $\Gamma^{\pm a} \otimes \Gamma^{\mp a}$ and $\Gamma^{\pm b} \otimes \Gamma^{\mp b}$ are multi-valued function. There are monodromy factors M^a and M^b respectively as following among different branches around $\lambda = \infty$

$$M^a = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j \geq 0} \lambda^j (s_j \otimes 1 - 1 \otimes s_j) \right\}, \quad (6.1)$$

$$M^b = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j \geq 0} \lambda^{-j} (s_j \otimes 1 - 1 \otimes s_j) \right\}. \quad (6.2)$$

In order to offset the complication, we need to generalize the concept of vertex operators which leads it to be not scalar-valued any more but take values in a differential operator algebra. So we introduce the following vertex operators

$$\Gamma_a^{\delta} = \exp \left(- \sum_{k=1}^N E_{kk} \sum_{j > 0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right) \exp(x \partial_{s_0}), \quad (6.3)$$

$$\Gamma_b^{\delta} = \exp \left(- \sum_{k=1}^N E_{kk} \sum_{j > 0} \frac{\lambda^{-j}}{\epsilon j} (\epsilon \partial_x) s_j \right) \exp(x \partial_{s_0}), \quad (6.4)$$

$$\Gamma_a^{\delta\#} = \exp(x \partial_{s_0}) \exp \left(\sum_{k=1}^N E_{kk} \sum_{j > 0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right), \quad (6.5)$$

$$\Gamma_b^{\delta\#} = \exp(x \partial_{s_0}) \exp \left(\sum_{k=1}^N E_{kk} \sum_{j > 0} \frac{\lambda^{-j}}{\epsilon j} (\epsilon \partial_x) s_j \right). \quad (6.6)$$

Then

$$\Gamma_a^{\delta\#} \otimes \Gamma_a^{\delta} = \exp(x \partial_{s_0}) \exp \left(\sum_{j > 0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) (s_j - s'_j) \right) \exp(x \partial_{s'_0}), \quad (6.7)$$

$$\Gamma_b^{\delta\#} \otimes \Gamma_b^{\delta} = \exp(x \partial_{s_0}) \exp \left(\sum_{j > 0} \frac{\lambda^{-j}}{\epsilon j} (\epsilon \partial_x) (s_j - s'_j) \right) \exp(x \partial_{s'_0}). \quad (6.8)$$

After computation we get

$$\begin{aligned} & \left(\Gamma_a^{\delta\#} \otimes \Gamma_a^{\delta} \right) M^a = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j > 0} \frac{\lambda^j}{j} (s_j - s'_j) \right\} \\ & \exp \left(\pm \frac{2\pi i}{\epsilon} ((s_0 + x) - (s'_0 + x + \sum_{j > 0} \frac{\lambda^j}{j} (s_j - s'_j))) \right) \left(\Gamma_a^{\delta\#} \otimes \Gamma_a^{\delta} \right) \\ & = \exp \left(\pm \frac{2\pi i}{\epsilon} (s_0 - s'_0) \right) \left(\Gamma_a^{\delta\#} \otimes \Gamma_a^{\delta} \right), \end{aligned}$$

$$\begin{aligned}
& \left(\Gamma_b^{\delta\#} \otimes \Gamma_b^\delta \right) M^b = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j>0} \frac{\lambda^{-j}}{j} (s_j - s'_j) \right\} \\
& \exp \left(\pm \frac{2\pi i}{\epsilon} ((s_0 + x) - \sum_{k=1}^N E_{kk} (s'_0 + x + \sum_{j>0} \frac{\lambda^{-j}}{j} (s_j - s'_j))) \right) \left(\Gamma_b^{\delta\#} \otimes \Gamma_b^\delta \right) \\
& = \exp \left(\pm \frac{2\pi i}{\epsilon} (s_0 - s'_0) \right) \left(\Gamma_b^{\delta\#} \otimes \Gamma_b^\delta \right).
\end{aligned}$$

Thus when $s_0 - s'_0 \in \mathbb{Z}\epsilon$, $\left(\Gamma_a^{\delta\#} \otimes \Gamma_a^\delta \right) (\Gamma^a \otimes \Gamma^{-a})$ and $\left(\Gamma_b^{\delta\#} \otimes \Gamma_b^\delta \right) (\Gamma^{-b} \otimes \Gamma^b)$ are all single-valued near $\lambda = \infty$.

Now we should note that the above vertex operators take value in differential operator algebra $\mathbb{C}[\partial, x, t, \bar{t}, s, \epsilon] := \{f(x, t, \epsilon) | f(x, t, \bar{t}, s, \epsilon) = \sum_{i \geq 0} c_i(x, t, \bar{t}, s, \epsilon) \partial^i\}$.

Theorem 1. *Function $\tau_M, \bar{\tau}_M$ are tau-functions of the EMTH if and only if they satisfy the following Hirota quadratic equations of the EMTH*

$$\text{Res}_\lambda \left(\lambda^{r-1} \left(\Gamma_a^{\delta\#} \otimes \Gamma_a^\delta \right) (\Gamma^a \otimes \Gamma^{-a}) (\tau_M \otimes \tau_M) - \lambda^{-r-1} \left(\Gamma_b^{\delta\#} \otimes \Gamma_b^\delta \right) (\Gamma^{-b} \otimes \Gamma^b) (\bar{\tau}_M \otimes \bar{\tau}_M) \right) = 0 \quad (6.9)$$

computed at $s_0 - s'_0 = l\epsilon$ for each $l \in \mathbb{Z}$, $r \in \mathbb{N}$.

Proof. We just need to prove that the HBEs are equivalent to the right side in Proposition 5. By a straightforward computation we can get the following four identities

$$\Gamma_a^{\delta\#} \Gamma^a \tau_M = \tau(s_0 + x - \frac{\epsilon}{2}, t, \bar{t}, s) \lambda^{\sum_{k=1}^N E_{kk} s_0 / \epsilon} \mathcal{W}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \lambda^{\mathbb{I}_N x / \epsilon}, \quad (6.10)$$

$$\Gamma_a^\delta \Gamma^{-a} \tau_M = \lambda^{-(s_0 + x) / \epsilon} \mathcal{W}^{-1}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \tau(x + s_0 + \frac{\epsilon}{2}, t, \bar{t}, s), \quad (6.11)$$

$$\Gamma_b^{\delta\#} \Gamma^{-b} \bar{\tau}_M = \tau(x + s_0 - \frac{\epsilon}{2}, t, \bar{t}, s) \lambda^{\sum_{k=1}^N E_{kk} s_0 / \epsilon} \bar{\mathcal{W}}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \lambda^{x \mathbb{I}_N / \epsilon}, \quad (6.12)$$

$$\Gamma_b^\delta \Gamma^b \bar{\tau}_M = \lambda^{-\sum_{k=1}^N E_{kk} s_0 / \epsilon} \lambda^{-x \mathbb{I}_N / \epsilon} \bar{\mathcal{W}}^{-1}(x, t, \bar{t}, s, \epsilon \partial_x, \lambda) \tau(x + s_0 + \frac{\epsilon}{2}, t, \bar{t}, s). \quad (6.13)$$

As an example, we only give the proof for the eq.(6.11),

$$\begin{aligned}
& \Gamma_a^\delta \Gamma^{-a} \tau_M \\
&= \exp \left(- \sum_{k=1}^N E_{kk} \sum_{j>0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right) \exp(x \partial_{s_0}) \\
& \quad \exp \left(- \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \left(\sum_{j=0}^{\infty} t_{jk} \lambda^j + s_j \lambda^j \log \lambda \right) \right) \times \exp \left(\frac{\epsilon}{2} \partial_{s_0} + [\lambda^{-1}]_{\partial} \right) \tau_M \\
&= \exp \left\{ \left[- \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j=0}^{\infty} t_{jk} \lambda^j - \sum_{k=1}^N E_{kk} \sum_{j>0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right] \right\} \\
& \quad \exp \left(- \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j>0} s_j \lambda^j \log \lambda \right) \exp \{ - (\log \lambda) (s_0 + x) / \epsilon \} \\
& \quad \tau_M(x + s_0 + \frac{\epsilon}{2}, t + [\lambda^{-1}], \bar{t}, s) \\
&= \lambda^{-(s_0+x)/\epsilon} \exp \left\{ \left[- \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j=0}^{\infty} t_{jk} \lambda^j - \sum_{k=1}^N E_{kk} \sum_{j>0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right] \right\} \\
& \quad \tau_M(x + s_0 + \frac{\epsilon}{2}, t + [\lambda^{-1}], \bar{t}, s) \\
&= \lambda^{-(s_0+x)/\epsilon} \exp \left\{ \left[- \frac{1}{\epsilon} \sum_{k=1}^N E_{kk} \sum_{j=0}^{\infty} t_{jk} \lambda^j - \sum_{k=1}^N E_{kk} \sum_{j>0} \frac{\lambda^j}{\epsilon j} (\epsilon \partial_x) s_j \right] \right\} \mathbb{S}^{-1}(x, t, \bar{t}, \lambda) \\
& \quad \tau(x + s_0 + \frac{\epsilon}{2}, t, \bar{t}, s) \\
&= \lambda^{-(s_0+x)/\epsilon} \mathcal{W}^{-1}(x, t, \epsilon \partial_x, \lambda) \tau(x + s_0 + \frac{\epsilon}{2}, t, \bar{t}, s).
\end{aligned}$$

So eq.(6.11) is proved. Eq.(6.10),(6.12),eq.(6.13) can be proved in similar ways. By substituting four equations eq.(6.10)-eq.(6.13) into the HBEs (6.9), eq.(4.10) is derived. \square

The eq.(6.9) in the case when $N = 1$ is exactly the Hirota quadratic equation of the extended Toda hierarchy in [7]. To give more information on the relations among different solutions of the EMTH, the Darboux transformation of the EMTH will be constructed using kernel determinant technique as [23, 24] in the next section.

7 Darboux transformations of the EMTH

In this section, we will consider the Darboux transformations of the EMTH on the Lax operator

$$\mathcal{L} = \Lambda + u + v \Lambda^{-1}, \quad (7.1)$$

i.e.

$$\mathcal{L}^{[1]} = \Lambda + u^{[1]} + v^{[1]} \Lambda^{-1} = W \mathcal{L} W^{-1}, \quad (7.2)$$

where W is the Darboux transformation operator.

That means after the Darboux transformation, the spectral problem about $N \times N$ spectral matrix ϕ

$$\mathcal{L}\phi = \Lambda\phi + u\phi + v\Lambda^{-1}\phi = \lambda\phi, \quad (7.3)$$

will become

$$\mathcal{L}^{[1]}\phi^{[1]} = \lambda\phi^{[1]}. \quad (7.4)$$

To keep the Lax pair of the EMTH invariant, i.e.

$$\epsilon\partial_{t_{jk}}\mathcal{L}^{[1]} = [(B_{jk}^{[1]})_+, \mathcal{L}^{[1]}], \quad \epsilon\partial_{t_{jk}}C_{ss}^{[1]} = [(B_{jk}^{[1]})_+, C_{ss}^{[1]}], \quad \epsilon\partial_{t_{jk}}\bar{C}_{ss}^{[1]} = [(B_{jk}^{[1]})_+, \bar{C}_{ss}^{[1]}], \quad (7.5)$$

$$\epsilon\partial_{\bar{t}_{jk}}\mathcal{L}^{[1]} = [(\bar{B}_{jk}^{[1]})_+, \mathcal{L}^{[1]}], \quad \epsilon\partial_{\bar{t}_{jk}}C_{ss}^{[1]} = [(\bar{B}_{jk}^{[1]})_+, C_{ss}^{[1]}], \quad \epsilon\partial_{\bar{t}_{jk}}\bar{C}_{ss}^{[1]} = [(\bar{B}_{jk}^{[1]})_+, \bar{C}_{ss}^{[1]}], \quad (7.6)$$

$$\epsilon\partial_{s_j}\mathcal{L}^{[1]} = [(D_j^{[1]})_+, \mathcal{L}^{[1]}], \quad \epsilon\partial_{s_j}C_{ss}^{[1]} = [(D_j^{[1]})_+, C_{ss}^{[1]}], \quad \epsilon\partial_{s_j}\bar{C}_{ss}^{[1]} = [(D_j^{[1]})_+, \bar{C}_{ss}^{[1]}], \quad (7.7)$$

$$\epsilon\partial_{t_{jk}}\log\mathcal{L}^{[1]} = [(B_{jk}^{[1]})_+, \log\mathcal{L}^{[1]}], \quad \epsilon\partial_{\bar{t}_{jk}}\log\mathcal{L}^{[1]} = [-(\bar{B}_{jk}^{[1]})_-, \log\mathcal{L}^{[1]}], \quad (7.8)$$

$$\epsilon(\log\mathcal{L}^{[1]})_{s_j} = [-(D_j^{[1]})_-, \frac{1}{2}\log_+\mathcal{L}^{[1]}] + [(D_j^{[1]})_+, \frac{1}{2}\log_-\mathcal{L}^{[1]}], \quad (7.9)$$

$$B_{\alpha,n}^{[1]} := B_{\alpha,n}(\mathcal{L}^{[1]}), \quad \bar{B}_{\alpha,n}^{[1]} := \bar{B}_{\alpha,n}(\mathcal{L}^{[1]}), \quad (7.10)$$

the dressing operator W should satisfy the following dressing equations

$$\epsilon W_{t_{j,n}} = -W(B_{j,n})_+ + (WB_{j,n}W^{-1})_+W, \quad 1 \leq n \leq N, j \geq 0, \quad (7.11)$$

$$\epsilon W_{\bar{t}_{j,n}} = -W(B_{j,n})_+ + (WB_{j,n}W^{-1})_+W, \quad 1 \leq n \leq N, j \geq 0, \quad (7.12)$$

$$\epsilon W_{s_j} = -W(D_j)_+ + (WD_jW^{-1})_+W, \quad j \geq 0. \quad (7.13)$$

where $W_{t_{\gamma,n}}$ means the derivative of W by $t_{\gamma,n}$. For a local operator $B = \sum_{m=0}^{\infty} b_m(x)\Lambda^m$, we define $B^*(g(x)) = \sum_{m=0}^{\infty} \Lambda^{-m}(g(x)b_m(x))$. To give the Darboux transformation, we need the following lemma.

Lemma 4. *The operator $B := \sum_{n=0}^{\infty} b_n\Lambda^n$ is a non-negative matrix-valued difference operator, $C := \sum_{n=1}^{\infty} c_n\Lambda^{-n}$ is a negative matrix-valued difference operator and f, g (short for $f(x), g(x)$) are two functions of spatial parameter x , following identities hold*

$$(Bf\frac{\Lambda^{-1}}{1-\Lambda^{-1}}g)_- = B(f)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}g, \quad (f\frac{\Lambda^{-1}}{1-\Lambda^{-1}}gB)_- = f\frac{\Lambda^{-1}}{1-\Lambda^{-1}}B^*(g), \quad (7.14)$$

$$(Cf\frac{1}{1-\Lambda}g)_+ = C(f)\frac{1}{1-\Lambda}g, \quad (f\frac{1}{1-\Lambda}gC)_+ = f\frac{1}{1-\Lambda}C^*(g). \quad (7.15)$$

Proof. Here we only give the proof of the eq.(7.14) by direct calculation

$$\begin{aligned}
(Bf \frac{\Lambda^{-1}}{1-\Lambda^{-1}}g)_- &= \sum_{m=0}^{\infty} b_m(f(x+m\epsilon)\Lambda^m \frac{\Lambda^{-1}}{1-\Lambda^{-1}}g)_- \\
&= \sum_{m=0}^{\infty} b_m f(x+m\epsilon) (\frac{\Lambda^{m-1}}{1-\Lambda^{-1}})_- g \\
&= \sum_{m=0}^{\infty} b_m f(x+m\epsilon) \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g \\
&= B(f) \frac{\Lambda^{-1}}{1-\Lambda^{-1}} g,
\end{aligned} \tag{7.16}$$

$$\begin{aligned}
(f \frac{\Lambda^{-1}}{1-\Lambda^{-1}}gB)_- &= \sum_{m=0}^{\infty} (f \frac{\Lambda^{-1}}{1-\Lambda^{-1}}g b_m \Lambda^m)_- \\
&= \sum_{m=0}^{\infty} (f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} \Lambda^m g(x-m\epsilon) b_m(x-m\epsilon))_- \\
&= \sum_{m=0}^{\infty} f(\frac{\Lambda^{m-1}}{1-\Lambda^{-1}})_- g(x-m\epsilon) b_m(x-m\epsilon) \\
&= f \frac{\Lambda^{-1}}{1-\Lambda^{-1}} B^*(g).
\end{aligned}$$

The similar proof for the eq.(7.15) can be got easily. □

Now, we will give the following important theorem which will be used to generate new solutions from seed solutions.

Theorem 2. *If ϕ is the first wave function of the EMTH, the Darboux transformation operator of the EMTH*

$$W(\lambda) = (1 - \phi(\phi(x - \epsilon))^{-1} \Lambda^{-1}) = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}, \tag{7.17}$$

will generate new solutions $u^{[1]}, v^{[1]}$ from seed solutions u, v

$$u^{[1]} = u + (\Lambda - 1) \phi(\phi(x - \epsilon))^{-1}, \tag{7.18}$$

$$v^{[1]} = \phi(\phi(x - \epsilon))^{-1} (\Lambda^{-1} v) (\Lambda^{-2} \phi) (\Lambda^{-1} \phi)^{-1}. \tag{7.19}$$

Proof. In the following proof, using eq.(7.14) in the Lemma 4, a direct computation will

lead to the following

$$\begin{aligned}
\epsilon W_{t_{\gamma,n}} W^{-1} &= (\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1})_{t_{\gamma,n}} \phi \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= (((B_{\gamma,n})_+ \phi) \circ (1 - \Lambda^{-1}) \circ \phi^{-1}) \phi \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&\quad - \phi \circ (1 - \Lambda^{-1}) \circ ((B_{\gamma,n})_+ \phi) \phi^{-1} \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= ((B_{\gamma,n})_+ \phi) \phi^{-1} - \phi \circ (1 - \Lambda^{-1}) \circ ((B_{\gamma,n})_+ \phi) \phi^{-1} \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&= -(\phi \circ [(1 - \Lambda^{-1}) \cdot \phi^{-1}(x)((B_{\gamma,n}(x))_+ \cdot \phi(x))] \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_- \\
&= -(\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (B_{\gamma,n}(x))_+ \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_- \\
&= -\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ (B_{\gamma,n})_+(x) \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1} \\
&\quad + (\phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}(x) \circ B_{\gamma,n}(x) \circ \phi(x) \circ (1 - \Lambda^{-1})^{-1} \circ \phi^{-1})_+ \\
&= -W(B_{\gamma,n})_+ W^{-1} + (W B_{\gamma,n} W^{-1})_+.
\end{aligned}$$

Therefore

$$W = \phi \circ (1 - \Lambda^{-1}) \circ \phi^{-1}, \quad (7.20)$$

can be as a Darboux transformation of the EMTH. Eqs.(7.18)-(7.19) can be directly got from the eq.(7.2). \square

Here we define $\phi_i = \phi_i^{[0]} := \phi|_{\lambda=\lambda_i}$, then one can choose the specific one-fold Darboux transformation of the EMTH as following

$$W_1(\lambda_1) = \mathbb{I}_N - \phi_1(\phi_1(x - \epsilon))^{-1} \Lambda^{-1}. \quad (7.21)$$

Meanwhile, we can also get the Darboux transformation on wave function ϕ as following

$$\phi^{[1]} = (\mathbb{I}_N - \phi_1(\phi_1(x - \epsilon))^{-1} \Lambda^{-1}) \phi. \quad (7.22)$$

Then using iteration on the Darboux transformation, the j -th Darboux transformation from the $(j-1)$ -th solution is as

$$\phi^{[j]} = (\mathbb{I}_N - \frac{\phi_j^{[j-1]}}{\Lambda^{-1} \phi_j^{[j-1]}} \Lambda^{-1}) \phi^{[j-1]}, \quad (7.23)$$

$$u^{[j]} = u^{[j-1]} + (\Lambda - 1) \phi_j^{[j-1]} (\Lambda^{-1} \phi_j^{[j-1]})^{-1}, \quad (7.24)$$

$$v^{[j]} = \phi_j^{[j-1]} (\Lambda^{-1} \phi_j^{[j-1]})^{-1} (\Lambda^{-1} v^{[j-1]}) \Lambda^{-2} \phi_j^{[j-1]} (\Lambda^{-1} \phi_j^{[j-1]})^{-1}, \quad (7.25)$$

where $\phi_i^{[j-1]} := \phi^{[j-1]}|_{\lambda=\lambda_i}$, are wave functions corresponding to different spectrals with the $(j-1)$ -th solutions $u^{[j-1]}, v^{[j-1]}$. It can be checked that $\phi_i^{[j-1]} = 0$, $i = 1, 2, \dots, j-1$.

After iteration on the Darboux transformations, the following theorem about the two-fold Darboux transformation of the EMTH can be derived by a direct calculation.

Theorem 3. *The two-fold Darboux transformation of the EMTH is as following*

$$W_2 = \mathbb{I}_N + t_1^{[2]} \Lambda^{-1} + t_2^{[2]} \Lambda^{-2}, \quad (7.26)$$

where

$$t_1^{[2]} = (\phi_1 \phi_2(x-2\epsilon) - \phi_2 \phi_1(x-2\epsilon))(\phi_1(x-\epsilon) \phi_2(x-2\epsilon) - \phi_2(x-\epsilon) \phi_1(x-2\epsilon))^{-1}, \quad (7.27)$$

$$t_2^{[2]} = (\phi_1 \phi_2(x-\epsilon) - \phi_2 \phi_1(x-\epsilon))(\phi_2(x-2\epsilon) \phi_1(x-\epsilon) - \phi_1(x-2\epsilon) \phi_2(x-\epsilon))^{-1}. \quad (7.28)$$

The Darboux transformation leads to new solutions from seed solutions

$$u^{[2]} = u + (\Lambda - 1)t_1^{[2]}, \quad (7.29)$$

$$v^{[2]} = t_2^{[2]}(x)(\Lambda^{-2}v)t_2^{[2]-1}(x-\epsilon). \quad (7.30)$$

In fact the Darboux operator W_2 can be further written as

$$(W_2)_{ij} = \frac{1}{\Delta_2} \begin{vmatrix} \delta_{ij} & 0 & \dots & \Lambda^{-1} & \dots & 0 & 0 & \dots & \Lambda^{-2} & \dots & 0 \\ \phi_{1,i1}(x) & \phi_{1,i1}(x-\epsilon) & \dots & \phi_{1,i1}(x-\epsilon) & \dots & \phi_{1,N1}(x-\epsilon) & \phi_{1,i1}(x-2\epsilon) & \dots & \phi_{1,i1}(x-2\epsilon) & \dots & \phi_{1,N1}(x-2\epsilon) \\ \phi_{1,i2}(x) & \phi_{1,i2}(x-\epsilon) & \dots & \phi_{1,i2}(x-\epsilon) & \dots & \phi_{1,N2}(x-\epsilon) & \phi_{1,i2}(x-2\epsilon) & \dots & \phi_{1,i2}(x-2\epsilon) & \dots & \phi_{1,N2}(x-2\epsilon) \\ \phi_{1,ii}(x) & \phi_{1,ii}(x-\epsilon) & \dots & \phi_{1,ii}(x-\epsilon) & \dots & \phi_{1,Ni}(x-\epsilon) & \phi_{1,ii}(x-2\epsilon) & \dots & \phi_{1,ii}(x-2\epsilon) & \dots & \phi_{1,Ni}(x-2\epsilon) \\ \phi_{1,iN}(x) & \phi_{1,iN}(x-\epsilon) & \dots & \phi_{1,iN}(x-\epsilon) & \dots & \phi_{1,NN}(x-\epsilon) & \phi_{1,iN}(x-2\epsilon) & \dots & \phi_{1,iN}(x-2\epsilon) & \dots & \phi_{1,NN}(x-2\epsilon) \\ \phi_{2,i1}(x) & \phi_{2,i1}(x-\epsilon) & \dots & \phi_{2,i1}(x-\epsilon) & \dots & \phi_{2,N1}(x-\epsilon) & \phi_{2,i1}(x-2\epsilon) & \dots & \phi_{2,i1}(x-2\epsilon) & \dots & \phi_{2,N1}(x-2\epsilon) \\ \phi_{2,i2}(x) & \phi_{2,i2}(x-\epsilon) & \dots & \phi_{2,i2}(x-\epsilon) & \dots & \phi_{2,N2}(x-\epsilon) & \phi_{2,i2}(x-2\epsilon) & \dots & \phi_{2,i2}(x-2\epsilon) & \dots & \phi_{2,N2}(x-2\epsilon) \\ \phi_{2,ii}(x) & \phi_{2,ii}(x-\epsilon) & \dots & \phi_{2,ii}(x-\epsilon) & \dots & \phi_{2,Ni}(x-\epsilon) & \phi_{2,ii}(x-2\epsilon) & \dots & \phi_{2,ii}(x-2\epsilon) & \dots & \phi_{2,Ni}(x-2\epsilon) \\ \phi_{2,iN}(x) & \phi_{2,iN}(x-\epsilon) & \dots & \phi_{2,iN}(x-\epsilon) & \dots & \phi_{2,NN}(x-\epsilon) & \phi_{2,iN}(x-2\epsilon) & \dots & \phi_{2,iN}(x-2\epsilon) & \dots & \phi_{2,NN}(x-2\epsilon) \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \phi_{1,i1}(x-\epsilon) & \dots & \phi_{1,i1}(x-\epsilon) & \dots & \phi_{1,N1}(x-\epsilon) & \phi_{1,i1}(x-2\epsilon) & \dots & \phi_{1,i1}(x-2\epsilon) & \dots & \phi_{1,N1}(x-2\epsilon) \\ \phi_{1,i2}(x-\epsilon) & \dots & \phi_{1,i2}(x-\epsilon) & \dots & \phi_{1,N2}(x-\epsilon) & \phi_{1,i2}(x-2\epsilon) & \dots & \phi_{1,i2}(x-2\epsilon) & \dots & \phi_{1,N2}(x-2\epsilon) \\ \phi_{1,i3}(x-\epsilon) & \dots & \phi_{1,i3}(x-\epsilon) & \dots & \phi_{1,N3}(x-\epsilon) & \phi_{1,i3}(x-2\epsilon) & \dots & \phi_{1,i3}(x-2\epsilon) & \dots & \phi_{1,N3}(x-2\epsilon) \\ \phi_{1,iN}(x-\epsilon) & \dots & \phi_{1,iN}(x-\epsilon) & \dots & \phi_{1,NN}(x-\epsilon) & \phi_{1,iN}(x-2\epsilon) & \dots & \phi_{1,iN}(x-2\epsilon) & \dots & \phi_{1,NN}(x-2\epsilon) \\ \phi_{2,i1}(x-\epsilon) & \dots & \phi_{2,i1}(x-\epsilon) & \dots & \phi_{2,N1}(x-\epsilon) & \phi_{2,i1}(x-2\epsilon) & \dots & \phi_{2,i1}(x-2\epsilon) & \dots & \phi_{2,N1}(x-2\epsilon) \\ \phi_{2,i2}(x-\epsilon) & \dots & \phi_{2,i2}(x-\epsilon) & \dots & \phi_{2,N2}(x-\epsilon) & \phi_{2,i2}(x-2\epsilon) & \dots & \phi_{2,i2}(x-2\epsilon) & \dots & \phi_{2,N2}(x-2\epsilon) \\ \phi_{2,i3}(x-\epsilon) & \dots & \phi_{2,i3}(x-\epsilon) & \dots & \phi_{2,N3}(x-\epsilon) & \phi_{2,i3}(x-2\epsilon) & \dots & \phi_{2,i3}(x-2\epsilon) & \dots & \phi_{2,N3}(x-2\epsilon) \\ \phi_{2,iN}(x-\epsilon) & \dots & \phi_{2,iN}(x-\epsilon) & \dots & \phi_{2,NN}(x-\epsilon) & \phi_{2,iN}(x-2\epsilon) & \dots & \phi_{2,iN}(x-2\epsilon) & \dots & \phi_{2,NN}(x-2\epsilon) \end{vmatrix}. \quad (7.31)$$

Similarly, we can generalize the Darboux transformation to the n -fold case which is contained in the following theorem.

Theorem 4. *The n -fold Darboux transformation of EMTH equation is as follows*

$$W_n = \mathbb{I}_N + t_1^{[n]} \Lambda^{-1} + t_2^{[n]} \Lambda^{-2} + \dots + t_n^{[n]} \Lambda^{-n} \quad (7.32)$$

where

$$W_n \cdot \phi_i|_{i \leq n} = 0. \quad (7.33)$$

The Darboux transformation leads to new solutions from seed solutions

$$u^{[n]} = u + (\Lambda - 1)t_1^{[n]}, \quad (7.34)$$

$$v^{[n]} = t_n^{[n]}(x)(\Lambda^{-n}v)t_n^{[n]-1}(x-\epsilon), \quad (7.35)$$

where

$$(W_n)_{ij} = \frac{1}{\Delta_n} \begin{vmatrix} \delta_{ij} & 0 & \dots & \Lambda^{-1} & \dots & 0 & 0 & \dots & 0 & \dots & \Lambda^{-n} & \dots & 0 \\ \phi_{1,i1}(x) & \phi_{1,11}(x-\epsilon) & \dots & \phi_{1,j1}(x-\epsilon) & \dots & \phi_{1,N1}(x-\epsilon) & \phi_{1,11}(x-2\epsilon) & \dots & \phi_{1,N1}(x-2\epsilon) & \dots & \phi_{1,j1}(x-n\epsilon) & \dots & \phi_{1,N1}(x-n\epsilon) \\ \phi_{1,i2}(x) & \phi_{1,12}(x-\epsilon) & \dots & \phi_{1,j2}(x-\epsilon) & \dots & \phi_{1,N2}(x-\epsilon) & \phi_{1,12}(x-2\epsilon) & \dots & \phi_{1,N2}(x-2\epsilon) & \dots & \phi_{1,j2}(x-n\epsilon) & \dots & \phi_{1,N2}(x-n\epsilon) \\ \phi_{1,ii}(x) & \phi_{1,1i}(x-\epsilon) & \dots & \phi_{1,ji}(x-\epsilon) & \dots & \phi_{1,Ni}(x-\epsilon) & \phi_{1,1i}(x-2\epsilon) & \dots & \phi_{1,Ni}(x-2\epsilon) & \dots & \phi_{1,ji}(x-n\epsilon) & \dots & \phi_{1,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{1,iN}(x) & \phi_{1,1N}(x-\epsilon) & \dots & \phi_{1,jN}(x-\epsilon) & \dots & \phi_{1,NN}(x-\epsilon) & \phi_{1,1N}(x-2\epsilon) & \dots & \phi_{1,NN}(x-2\epsilon) & \dots & \phi_{1,jN}(x-n\epsilon) & \dots & \phi_{1,NN}(x-n\epsilon) \\ \phi_{2,i1}(x) & \phi_{2,11}(x-\epsilon) & \dots & \phi_{2,j1}(x-\epsilon) & \dots & \phi_{2,N1}(x-\epsilon) & \phi_{2,21}(x-2\epsilon) & \dots & \phi_{2,N1}(x-2\epsilon) & \dots & \phi_{2,j1}(x-n\epsilon) & \dots & \phi_{2,N1}(x-n\epsilon) \\ \phi_{2,i2}(x) & \phi_{2,12}(x-\epsilon) & \dots & \phi_{2,j2}(x-\epsilon) & \dots & \phi_{2,N2}(x-\epsilon) & \phi_{2,12}(x-2\epsilon) & \dots & \phi_{2,N2}(x-2\epsilon) & \dots & \phi_{2,j2}(x-n\epsilon) & \dots & \phi_{2,N2}(x-n\epsilon) \\ \phi_{2,ii}(x) & \phi_{2,1i}(x-\epsilon) & \dots & \phi_{2,ji}(x-\epsilon) & \dots & \phi_{2,Ni}(x-\epsilon) & \phi_{2,1i}(x-2\epsilon) & \dots & \phi_{2,Ni}(x-2\epsilon) & \dots & \phi_{2,ji}(x-n\epsilon) & \dots & \phi_{2,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{2,iN}(x) & \phi_{2,1N}(x-\epsilon) & \dots & \phi_{2,jN}(x-\epsilon) & \dots & \phi_{2,NN}(x-\epsilon) & \phi_{2,1N}(x-2\epsilon) & \dots & \phi_{2,NN}(x-2\epsilon) & \dots & \phi_{2,jN}(x-n\epsilon) & \dots & \phi_{2,NN}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{n,i1}(x) & \phi_{n,11}(x-\epsilon) & \dots & \phi_{n,j1}(x-\epsilon) & \dots & \phi_{n,N1}(x-\epsilon) & \phi_{n,21}(x-2\epsilon) & \dots & \phi_{n,N1}(x-2\epsilon) & \dots & \phi_{n,j1}(x-n\epsilon) & \dots & \phi_{n,N1}(x-n\epsilon) \\ \phi_{n,i2}(x) & \phi_{n,12}(x-\epsilon) & \dots & \phi_{n,j2}(x-\epsilon) & \dots & \phi_{n,N2}(x-\epsilon) & \phi_{n,12}(x-2\epsilon) & \dots & \phi_{n,N2}(x-2\epsilon) & \dots & \phi_{n,j2}(x-n\epsilon) & \dots & \phi_{n,N2}(x-n\epsilon) \\ \phi_{n,ii}(x) & \phi_{n,1i}(x-\epsilon) & \dots & \phi_{n,ji}(x-\epsilon) & \dots & \phi_{n,Ni}(x-\epsilon) & \phi_{n,1i}(x-2\epsilon) & \dots & \phi_{n,Ni}(x-2\epsilon) & \dots & \phi_{n,ji}(x-n\epsilon) & \dots & \phi_{n,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{n,iN}(x) & \phi_{n,1N}(x-\epsilon) & \dots & \phi_{n,jN}(x-\epsilon) & \dots & \phi_{n,NN}(x-\epsilon) & \phi_{n,1N}(x-2\epsilon) & \dots & \phi_{n,NN}(x-2\epsilon) & \dots & \phi_{n,jN}(x-n\epsilon) & \dots & \phi_{n,NN}(x-n\epsilon) \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} \phi_{1,11}(x-\epsilon) & \phi_{1,21}(x-\epsilon) & \dots & \phi_{1,N1}(x-\epsilon) & \phi_{1,11}(x-2\epsilon) & \phi_{1,21}(x-2\epsilon) & \dots & \phi_{1,N1}(x-2\epsilon) & \phi_{1,21}(x-n\epsilon) & \dots & \phi_{1,N1}(x-n\epsilon) \\ \phi_{1,12}(x-\epsilon) & \phi_{1,22}(x-\epsilon) & \dots & \phi_{1,N2}(x-\epsilon) & \phi_{1,12}(x-2\epsilon) & \phi_{1,22}(x-2\epsilon) & \dots & \phi_{1,N2}(x-2\epsilon) & \phi_{1,22}(x-n\epsilon) & \dots & \phi_{1,N2}(x-n\epsilon) \\ \phi_{1,1i}(x-\epsilon) & \phi_{1,2i}(x-\epsilon) & \dots & \phi_{1,Ni}(x-\epsilon) & \phi_{1,1i}(x-2\epsilon) & \phi_{1,2i}(x-2\epsilon) & \dots & \phi_{1,Ni}(x-2\epsilon) & \phi_{1,2i}(x-n\epsilon) & \dots & \phi_{1,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{1,1N}(x-\epsilon) & \phi_{1,2N}(x-\epsilon) & \dots & \phi_{1,NN}(x-\epsilon) & \phi_{1,1N}(x-2\epsilon) & \phi_{1,2N}(x-2\epsilon) & \dots & \phi_{1,NN}(x-2\epsilon) & \phi_{1,2N}(x-n\epsilon) & \dots & \phi_{1,NN}(x-n\epsilon) \\ \phi_{2,11}(x-\epsilon) & \phi_{2,21}(x-\epsilon) & \dots & \phi_{2,N1}(x-\epsilon) & \phi_{2,21}(x-2\epsilon) & \phi_{2,21}(x-2\epsilon) & \dots & \phi_{2,N1}(x-2\epsilon) & \phi_{2,21}(x-n\epsilon) & \dots & \phi_{2,N1}(x-n\epsilon) \\ \phi_{2,12}(x-\epsilon) & \phi_{2,22}(x-\epsilon) & \dots & \phi_{2,N2}(x-\epsilon) & \phi_{2,12}(x-2\epsilon) & \phi_{2,22}(x-2\epsilon) & \dots & \phi_{2,N2}(x-2\epsilon) & \phi_{2,22}(x-n\epsilon) & \dots & \phi_{2,N2}(x-n\epsilon) \\ \phi_{2,1i}(x-\epsilon) & \phi_{2,2i}(x-\epsilon) & \dots & \phi_{2,Ni}(x-\epsilon) & \phi_{2,1i}(x-2\epsilon) & \phi_{2,2i}(x-2\epsilon) & \dots & \phi_{2,Ni}(x-2\epsilon) & \phi_{2,2i}(x-n\epsilon) & \dots & \phi_{2,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{2,1N}(x-\epsilon) & \phi_{2,2N}(x-\epsilon) & \dots & \phi_{2,NN}(x-\epsilon) & \phi_{2,1N}(x-2\epsilon) & \phi_{2,2N}(x-2\epsilon) & \dots & \phi_{2,NN}(x-2\epsilon) & \phi_{2,2N}(x-n\epsilon) & \dots & \phi_{2,NN}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n,11}(x-\epsilon) & \phi_{n,21}(x-\epsilon) & \dots & \phi_{n,N1}(x-\epsilon) & \phi_{n,21}(x-2\epsilon) & \phi_{n,21}(x-2\epsilon) & \dots & \phi_{n,N1}(x-2\epsilon) & \phi_{n,21}(x-n\epsilon) & \dots & \phi_{n,N1}(x-n\epsilon) \\ \phi_{n,12}(x-\epsilon) & \phi_{n,22}(x-\epsilon) & \dots & \phi_{n,N2}(x-\epsilon) & \phi_{n,12}(x-2\epsilon) & \phi_{n,22}(x-2\epsilon) & \dots & \phi_{n,N2}(x-2\epsilon) & \phi_{n,22}(x-n\epsilon) & \dots & \phi_{n,N2}(x-n\epsilon) \\ \phi_{n,1i}(x-\epsilon) & \phi_{n,2i}(x-\epsilon) & \dots & \phi_{n,Ni}(x-\epsilon) & \phi_{n,1i}(x-2\epsilon) & \phi_{n,2i}(x-2\epsilon) & \dots & \phi_{n,Ni}(x-2\epsilon) & \phi_{n,2i}(x-n\epsilon) & \dots & \phi_{n,Ni}(x-n\epsilon) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n,1N}(x-\epsilon) & \phi_{n,2N}(x-\epsilon) & \dots & \phi_{n,NN}(x-\epsilon) & \phi_{n,1N}(x-2\epsilon) & \phi_{n,2N}(x-2\epsilon) & \dots & \phi_{n,NN}(x-2\epsilon) & \phi_{n,2N}(x-n\epsilon) & \dots & \phi_{n,NN}(x-n\epsilon) \end{vmatrix}.$$

It can be easily checked that $W_n \phi_i = 0$, $i = 1, 2, \dots, n$.

8 bi-Hamiltonian structure and tau symmetry

To describe the integrability of the EMTH with the Lax operator

$$\mathcal{L} = \Lambda + u + v\Lambda^{-1}, \quad (8.1)$$

we will construct the bi-Hamiltonian structure and tau symmetry of the EMTH in this section. For a matrix $A = (a_{ij})$, the vector field ∂_A over EMTH is defined by

$$\partial_A = \sum_{i,j=1}^N \sum_{k \geq 0} a_{ij}^{(k)} \left(\frac{\partial}{\partial u_{ij}^{(k)}} + \frac{\partial}{\partial v_{ij}^{(k)}} \right) = \text{Tr} \sum_{k \geq 0} A^{(k)} \left(\frac{\partial}{\partial u^{(k)}} + \frac{\partial}{\partial v^{(k)}} \right), \quad (8.2)$$

where

$$\left(\frac{\partial}{\partial u^{(k)}} \right)_{ji} = \frac{\partial}{\partial u_{ij}^{(k)}}, \quad \left(\frac{\partial}{\partial v^{(k)}} \right)_{ji} = \frac{\partial}{\partial v_{ij}^{(k)}}. \quad (8.3)$$

For two functionals $\bar{f} = \int f dx$, $\bar{g} = \int g dx$, we have

$$\partial_A \bar{f} = \int \sum_{i,j=1}^N \sum_{k \geq 0} a_{ij}^{(k)} \left(\frac{\partial f}{\partial u_{ij}^{(k)}} + \frac{\partial f}{\partial v_{ij}^{(k)}} \right) dx = \int \text{Tr} \sum_{k \geq 0} A^{(k)} \left(\frac{\delta f}{\delta u^{(k)}} + \frac{\delta f}{\delta v^{(k)}} \right) dx. \quad (8.4)$$

Then we can define the Hamiltonian bracket as

$$\{\bar{f}, \bar{g}\} = \int \sum_{w, w'} \frac{\delta f}{\delta w} \{w, w'\} \frac{\delta g}{\delta w'} dx, \quad w, w' = u_{ij} \text{ or } v_{ij}, \quad 1 \leq i, j \leq N. \quad (8.5)$$

The bi-Hamiltonian structure for the EMTH can be given by the following two compatible Poisson brackets which is a generalization in matrix forms of the extended Toda hierarchy in [6]

$$\{u(x)_{ij}, u(y)_{pq}\}_1 = \frac{1}{\epsilon} [\delta_{iq} u_{pj}(x) - \delta_{jp} u_{iq}(x)] \delta(x - y), \quad (8.6)$$

$$\{u(x)_{ij}, v(y)_{pq}\}_1 = \frac{1}{\epsilon} [\delta_{iq} \Lambda v_{pj}(x) - \delta_{jp} v_{iq}(x)] \delta(x - y), \quad (8.7)$$

$$\{v(x)_{ij}, v(y)_{pq}\}_1 = 0, \quad (8.8)$$

$$\begin{aligned} \{u(x)_{ij}, u(y)_{pq}\}_2 &= \frac{1}{\epsilon} \left[\delta_{iq} \Lambda v_{pj}(x) - \delta_{jp} v_{iq}(x) \Lambda^{-1} + \delta_{iq} \sum_{s=1}^N u_{sj} \frac{\Lambda}{\Lambda - 1} u_{ps} - u_{pj} \frac{\Lambda}{\Lambda - 1} u_{iq} \right. \\ &\quad \left. - u_{iq} (\Lambda - 1)^{-1} u_{pj} + \delta_{jp} \sum_{s=1}^N u_{is} (\Lambda - 1)^{-1} u_{sq} \right] \delta(x - y), \end{aligned} \quad (8.9)$$

$$\begin{aligned} \{u(x)_{ij}, v(y)_{pq}\}_2 &= \frac{1}{\epsilon} \left[\delta_{iq} \sum_{s=1}^N u_{sj} \Lambda^2 (\Lambda - 1)^{-1} v_{ps}(x) - u_{pj} \Lambda (\Lambda - 1)^{-1} v_{iq} \right. \\ &\quad \left. - u_{iq} (\Lambda - 1)^{-1} \Lambda v_{pj}(x) + \delta_{jp} \sum_{s=1}^N u_{is} (\Lambda - 1)^{-1} v_{sq} \right] \delta(x - y), \end{aligned} \quad (8.10)$$

$$\begin{aligned} \{v(x)_{ij}, v(y)_{pq}\}_2 &= \frac{1}{\epsilon} \left[\delta_{iq} \sum_{s=1}^N v_{sj} \Lambda^2 (\Lambda - 1)^{-1} v_{ps}(x) - v_{pj} \Lambda (\Lambda - 1)^{-1} v_{iq} \right. \\ &\quad \left. - v_{iq} (\Lambda - 1)^{-1} v_{pj}(x) + \delta_{jp} \sum_{s=1}^N v_{is} \Lambda^{-1} (\Lambda - 1)^{-1} v_{sq} \right] \delta(x - y). \end{aligned} \quad (8.11)$$

One can check the anti-symmetric property of the above Poisson bracket, i.e.

$$\{u_{ij}, u_{pq}\}_i = -\{u_{pq}, u_{ij}\}_i, \quad \{v_{ij}, v_{pq}\} = -\{v_{pq}, v_{ij}\}_i, \quad i = 1, 2, \quad (8.12)$$

and so on. Further one can prove the two Poisson brackets all satisfying the anti-symmetric property. Also one can prove the complicated Jacobi identities over the two Hamiltonian structures about which the detail will not be given here explicitly. Also the compatibility of the two Hamiltonian structures can be got after the following theorem which shows the bi-Hamiltonian recursion relation.

When $N = 1$, the bi-Hamiltonian structure will be reduced to the following bracket

$$\{u(x), u(y)\}_1 = \{v(x), v(y)\}_1 = 0, \quad (8.13)$$

$$\{u(x), v(y)\}_1 = \frac{1}{\epsilon} [\Lambda v(x) - 1] \delta(x - y), \quad (8.14)$$

$$\{u(x), u(y)\}_2 = \frac{1}{\epsilon} [\Lambda v(x) - v(x) \Lambda^{-1}] \delta(x - y),$$

$$\{u(x), v(y)\}_2 = \frac{1}{\epsilon} [u(x) (\Lambda - 1) v(x)] \delta(x - y),$$

$$\{v(x), v(y)\}_2 = \frac{1}{\epsilon} [v(x) \Lambda v(x) - v(x) \Lambda^{-1} v(x)] \delta(x - y),$$

which can be rewritten as

$$\{u(x), u(y)\}_1 = \{\log v(x), \log v(y)\}_1 = 0, \quad (8.15)$$

$$\{u(x), \log v(y)\}_1 = \frac{1}{\epsilon} [\Lambda - 1] \delta(x - y), \quad (8.16)$$

$$\{u(x), u(y)\}_2 = \frac{1}{\epsilon} [\Lambda v(x) - v(x) \Lambda^{-1}] \delta(x - y),$$

$$\{u(x), \log v(y)\}_2 = \frac{1}{\epsilon} [u(x)(\Lambda - 1)] \delta(x - y),$$

$$\{\log v(x), \log v(y)\}_2 = \frac{1}{\epsilon} [\Lambda - \Lambda^{-1}] \delta(x - y).$$

In the above computation, the following identity is used

$$\Lambda[v(x)\delta(x - y)] = v(x + \epsilon)\delta(x + \epsilon - y) = v(y)\delta(x + \epsilon - y) = v(y)\Lambda\delta(x - y). \quad (8.17)$$

This is exactly the bi-Hamiltonian structure of the extended Toda hierarchy in [6] if we rewrite the $\log v$ equivalently to a new function u in [6].

For any difference operator $A = \sum_k A_k \Lambda^k$, we define its residue as $\text{Res } A = A_0$. In the following theorem, we will prove the above Poisson structures can be considered as the bi-Hamiltonian structure of the METH.

Theorem 5. *The flows of the EMTH are Hamiltonian systems of the form*

$$\frac{\partial u_{pq}}{\partial t_{j,k}} = \{u_{pq}, H_{j,k}\}_1, \quad \frac{\partial v_{pq}}{\partial t_{j,k}} = \{v_{pq}, H_{j,k}\}_1, \quad (8.18)$$

$$\frac{\partial u_{pq}}{\partial \bar{t}_{j,k}} = \{u_{pq}, \bar{H}_{j,k}\}_1, \quad \frac{\partial v_{pq}}{\partial \bar{t}_{j,k}} = \{v_{pq}, \bar{H}_{j,k}\}_1, \quad (8.19)$$

$$\frac{\partial u_{pq}}{\partial s_j} = \{u_{pq}, \tilde{H}_j\}_1, \quad \frac{\partial v_{pq}}{\partial s_j} = \{v_{pq}, \tilde{H}_j\}_1, \quad k = 0, 1, \dots, N; \quad j \geq 0. \quad (8.20)$$

They satisfy the following bi-Hamiltonian recursion relation

$$\{\cdot, H_{n-1,k}\}_2 = \{\cdot, H_{n,k}\}_1, \quad \{\cdot, \bar{H}_{n-1,k}\}_2 = \{\cdot, \bar{H}_{n,k}\}_1, \quad (8.21)$$

$$\{\cdot, \tilde{H}_{n-1}\}_2 = n\{\cdot, \tilde{H}_n\}_1 + \frac{2}{n!} \sum_{k=1}^N \{\cdot, H_{n,k}\}_1. \quad (8.22)$$

Here the Hamiltonians have the form

$$F_{j,k} = \int f_{j,k}(u, v; u_x, v_x; \dots; \epsilon) dx, \quad (8.23)$$

with the Hamiltonian $F_{j,k} = H_{j,k}, \bar{H}_{j,k}, \tilde{H}_j$ and the Hamiltonian densities $f_{j,k} = h_{j,k}, \bar{h}_{j,k}, \tilde{h}_j$ given by

$$h_{j,k} = \text{Tr Res } C_{kk} \mathcal{L}^j, \quad \bar{h}_{j,k} = \text{Tr Res } \bar{C}_{kk} \mathcal{L}^j, \quad (8.24)$$

$$\tilde{h}_j = \frac{2}{j!} \text{Tr Res } [\mathcal{L}^j (\log \mathcal{L} - c_j)]. \quad (8.25)$$

Proof. Here we will only prove that the flows $\frac{\partial}{\partial s_n}$ are Hamiltonian systems with respect to the first Poisson bracket and their recursion relation. The other cases can be proved in a more simple way. In [6], the following identity has been proved

$$TrRes [\mathcal{L}^n d(S\epsilon\partial_x S^{-1})] \sim TrRes \mathcal{L}^{n-1} d\mathcal{L}, \quad (8.26)$$

which show the validity of the following equivalence relation:

$$TrRes (\mathcal{L}^n d\log_+ \mathcal{L}) \sim TrRes (\mathcal{L}^{n-1} d\mathcal{L}). \quad (8.27)$$

Here the equivalent relation \sim is up to a x -derivative of another 1-form.

In a similar way as eq.(8.26), we obtain the following equivalence relation

$$TrRes [\mathcal{L}^n d(\bar{S}\epsilon\partial_x \bar{S}^{-1})] \sim -TrRes \mathcal{L}^{n-1} d\mathcal{L}, \quad (8.28)$$

i.e.

$$TrRes (\mathcal{L}^n d\log_- \mathcal{L}) \sim TrRes (\mathcal{L}^{n-1} d\mathcal{L}), \quad (8.29)$$

which further leads to

$$TrRes (\mathcal{L}^n d\log \mathcal{L}) \sim TrRes (\mathcal{L}^{n-1} d\mathcal{L}). \quad (8.30)$$

We suppose

$$D_n = \sum_k a_{n;k} \Lambda^k, \quad \mathcal{L}^n = \sum_j b_{n;j} \Lambda^j, \quad (8.31)$$

then from

$$\epsilon\partial_{s_n} \mathcal{L} = [(D_n)_+, \mathcal{L}] = [-(D_n)_-, \mathcal{L}], \quad (8.32)$$

we can derive equations

$$\epsilon \frac{\partial u}{\partial s_n} = a_{n;1}(x)v(x+\epsilon) - v(x)a_{n;1}(x-\epsilon) + [a_{n;0}(x), u(x)], \quad (8.33)$$

$$\epsilon \frac{\partial v}{\partial s_n} = a_{n;0}(x)v(x) - v(x)a_{n;0}(x-\epsilon). \quad (8.34)$$

The equivalence relation (8.30) now readily follows from the above two equations. By using (8.30) we obtain

$$\begin{aligned} d\tilde{h}_n &= \frac{2}{n!} d TrRes [\mathcal{L}^n (\log \mathcal{L} - c_n)] \\ &\sim \frac{2}{(n-1)!} TrRes [\mathcal{L}^{n-1} (\log \mathcal{L} - c_n) d\mathcal{L}] + \frac{2}{n!} TrRes [\mathcal{L}^{n-1} d\mathcal{L}] \\ &= \frac{2}{(n-1)!} TrRes [\mathcal{L}^{n-1} (\log \mathcal{L} - c_{n-1}) d\mathcal{L}] \end{aligned} \quad (8.35)$$

$$= Tr [a_{n-1;0}(x)du + a_{n-1;1}(x)dv(x+\epsilon)]. \quad (8.36)$$

Then we have

$$\begin{aligned} d\tilde{H}_n &= \frac{2}{n!} d \int TrRes [\mathcal{L}^n (\log \mathcal{L} - c_n)] \\ &\sim \int Tr [a_{n-1;0}(x)du + a_{n-1;1}(x)dv(x+\epsilon)] \end{aligned} \quad (8.37)$$

$$:= \int Tr \left[\frac{\delta \tilde{H}_n}{\delta u} du + \frac{\delta \tilde{H}_n}{\delta v} dv \right]. \quad (8.38)$$

It yields the following identities

$$\frac{\delta \tilde{H}_n}{\delta u_{ij}} = a_{n-1;0}(x)_{ji}, \quad \frac{\delta \tilde{H}_n}{\delta v_{ij}} = a_{n-1;1}(x-\epsilon)_{ji}. \quad (8.39)$$

This agrees with the Lax equation

$$\begin{aligned} \frac{\partial u_{ij}}{\partial s_n} &= \{u_{ij}, \tilde{H}_{n+1}\}_1 \\ &= \sum_{p,q=1}^N \frac{1}{\epsilon} [\delta_{iq} u_{pj}(x) - \delta_{jp} u_{iq}(x)] \frac{\delta \tilde{H}_{n+1}}{\delta u_{pq}} + \sum_{p,q=1}^N \frac{1}{\epsilon} [\delta_{iq} \Lambda v_{pj}(x) - \delta_{jp} v_{iq}(x)] \frac{\delta \tilde{H}_{n+1}}{\delta v_{pq}} \\ &= \frac{1}{\epsilon} \left[\sum_{p=1}^N a_{n,0}(x)_{ip} u_{pj}(x) - \sum_{q=1}^N u_{iq}(x) a_{n,0}(x)_{qj} \right] \\ &\quad + \frac{1}{\epsilon} \left[\sum_{p=1}^N a_{n,1}(x)_{ip} v(x+\epsilon)_{pj} - \sum_{q=1}^N v(x)_{iq} a_{n,1}(x-\epsilon)_{qj} \right] \\ &= \frac{1}{\epsilon} [a_{n,0}(x), u(x)]_{ij} + \frac{1}{\epsilon} [a_{n,1}(x)v(x+\epsilon) - v(x)a_{n,1}(x-\epsilon)]_{ij}, \end{aligned} \quad (8.40)$$

$$\begin{aligned} \frac{\partial v_{ij}}{\partial s_n} &= \{v_{ij}, \tilde{H}_{n+1}\}_1 = - \sum_{p,q=1}^N \frac{1}{\epsilon} [\delta_{pj} v_{iq}(x) \Lambda^{-1} - \delta_{qi} v_{pj}(x)] \frac{\delta \tilde{H}_{n+1}}{\delta u_{pq}} \\ &= \frac{1}{\epsilon} \left[\sum_{p=1}^N a_{n,0}(x)_{ip} v(x)_{pj} - \sum_{q=1}^N v(x)_{iq} a_{n,0}(x-\epsilon)_{qj} \right] \\ &= \frac{1}{\epsilon} [a_{n,0}(x)v(x) - v(x)a_{n,0}(x-\epsilon)]_{ij}. \end{aligned} \quad (8.41)$$

From the above identities we see that the flows $\frac{\partial}{\partial s_n}$ are Hamiltonian systems of the form (8.20). The recursion relation (8.22) follows from the following trivial identities

$$\begin{aligned} n \frac{2}{n!} \mathcal{L}^n (\log \mathcal{L} - c_n) &= \mathcal{L} \frac{2}{(n-1)!} \mathcal{L}^{n-1} (\log \mathcal{L} - c_{n-1}) - 2 \frac{1}{n!} \mathcal{L}^n \\ &= \frac{2}{(n-1)!} \mathcal{L}^{n-1} (\log \mathcal{L} - c_{n-1}) \mathcal{L} - 2 \frac{1}{n!} \mathcal{L}^n. \end{aligned}$$

Then we get,

$$\begin{aligned} n a_{n;0}(x) &= a_{n-1;-1}(x+\epsilon) + u(x) a_{n-1;0}(x) + v(x) a_{n-1;1}(x-\epsilon) - \frac{2}{n!} b_{n;0}(x) \\ &= a_{n-1;-1}(x) + a_{n-1;0}(x) u(x) + a_{n-1;1}(x) v(x+\epsilon) - \frac{2}{n!} b_{n;0}(x), \end{aligned} \quad (8.42)$$

$$\begin{aligned} n a_{n;1}(x) &= a_{n-1;0}(x+\epsilon) + u(x) a_{n-1;1}(x) + v(x) a_{n-1;2}(x-\epsilon) - \frac{2}{n!} b_{n;1}(x) \\ &= a_{n-1;0}(x) + a_{n-1;1}(x) u(x+\epsilon) + a_{n-1;2}(x) v(x+2\epsilon) - \frac{2}{n!} b_{n;1}(x). \end{aligned} \quad (8.43)$$

From eq.(8.42), we can derive the following equation by which one can represent $a_{n-1;-1}(x)$ in terms of other functions

$$(\Lambda - 1) a_{n-1;-1}(x) = a_{n-1;1}(x) v(x+\epsilon) - v(x) a_{n-1;1}(x-\epsilon) + [a_{n-1;0}(x), u(x)].$$

This further leads to the following recursion relation between two Poisson brackets

$$\begin{aligned}
& \{u_{ij}, \tilde{H}_n\}_2 \\
&= \sum_{p,q,s=1}^N \frac{1}{\epsilon} [\delta_{iq}\Lambda v_{pj}(x) - \delta_{jp}v_{iq}(x)\Lambda^{-1} + \delta_{iq}u_{sj}\Lambda(\Lambda-1)^{-1}u_{ps} - u_{pj}\Lambda(\Lambda-1)^{-1}u_{iq} \\
&\quad - u_{iq}(\Lambda-1)^{-1}u_{pj} + \delta_{jp}u_{is}(\Lambda-1)^{-1}u_{sq}] a_{n-1;0}(x)_{qp} \\
&\quad + \sum_{p,q,s=1}^N \frac{1}{\epsilon} [\delta_{iq}u_{sj}\Lambda(\Lambda-1)^{-1}v_{ps}(x+\epsilon)\Lambda - u_{pj}\Lambda(\Lambda-1)^{-1}v_{iq} \\
&\quad - u_{iq}(\Lambda-1)^{-1}v_{pj}(x+\epsilon)\Lambda + \delta_{jp}u_{is}(\Lambda-1)^{-1}v_{sq}] a_{n-1;1}(x-\epsilon)_{qp} \\
&= \frac{1}{\epsilon} \{a_{n-1;0}(x+\epsilon)v(x+\epsilon) - v(x)a_{n-1;0}(x-\epsilon) \\
&\quad + \Lambda(\Lambda-1)^{-1}(a_{n-1;1}(x)v(x+\epsilon) - v(x)a_{n-1;1}(x-\epsilon) + [a_{n-1;0}(x), u(x)]u(x) \\
&\quad - u(x)(\Lambda-1)^{-1}(a_{n-1;1}(x)v(x+\epsilon) - v(x)a_{n-1;1}(x-\epsilon) + [a_{n-1;0}(x), u(x)])\}_{ij} \\
&= \frac{1}{\epsilon} \{a_{n-1;0}(x+\epsilon)v(x+\epsilon) - v(x)a_{n-1;0}(x-\epsilon) + u(x)a_{n-1;1}(x)v(x+\epsilon) - v(x)a_{n-1;1}(x-\epsilon)u(x) \\
&\quad + a_{n-1;-1}(x+\epsilon)u(x) - u(x)a_{n-1;-1}(x) + v(x)a_{n-1;1}(x-\epsilon)u(x) - u(x)a_{n-1;1}(x)v(x+\epsilon)\}_{ij} \\
&= \frac{n}{\epsilon} [a_{n;1}(x)v(x+\epsilon) - v(x)a_{n;1}(x-\epsilon) + [a_{n;0}(x), u(x)]]_{ij} \\
&\quad + \frac{2}{\epsilon n!} [b_{n;1}(x)v(x+\epsilon) - v(x)b_{n;1}(x-\epsilon) + [b_{n;0}(x), u(x)]]_{ij} \\
&= n\{u_{ij}, \tilde{H}_n\}_1 + \frac{2}{n!} \sum_{k=1}^N \{u_{ij}, H_{n,k}\}_1.
\end{aligned}$$

This is exactly the recursion relation eq.(8.22) for matrix u . The similar recursion flow

on the matrix function v can be similarly derived by the following calculation,

$$\begin{aligned}
& \{v_{ij}, \tilde{H}_n\}_2 \\
&= \sum_{p,q,s=1}^N \frac{1}{\epsilon} [\delta_{iq} v_{sj} \Lambda^2 (\Lambda - 1)^{-1} v_{ps}(x) - v_{pj} \Lambda (\Lambda - 1)^{-1} v_{iq} \\
&\quad - v_{iq} (\Lambda - 1)^{-1} v_{pj}(x) + \delta_{jp} v_{is} \Lambda^{-1} (\Lambda - 1)^{-1} v_{sq}] a_{n-1;1}(x - \epsilon)_{qp} \\
&\quad + \sum_{p,q,s=1}^N \frac{1}{\epsilon} [u_{iq}(x) v_{pj}(x) - v_{iq}(x) \Lambda^{-1} u_{pj}(x) + \delta_{iq} v_{sj}(x) \Lambda (\Lambda - 1)^{-1} u_{ps} \\
&\quad - v_{pj}(x) \Lambda (\Lambda - 1)^{-1} u_{iq} - v_{iq} \Lambda^{-1} (\Lambda - 1)^{-1} u_{pj} + \delta_{pj} v_{is} \Lambda^{-1} (\Lambda - 1)^{-1} u_{sq}] a_{n-1;0}(x)_{qp} \\
&= \frac{1}{\epsilon} \{u(x) a_{n-1;0}(x) v(x) - v(x) a_{n-1;0}(x - \epsilon) u(x - \epsilon) \\
&\quad + [\Lambda (\Lambda - 1)^{-1} (a_{n-1;1}(x) v(x + \epsilon) - v(x) a_{n-1;1}(x - \epsilon) + [a_{n-1;0}(x), u(x)])] v(x) \\
&\quad - v(x) \Lambda^{-1} (\Lambda - 1)^{-1} (a_{n-1;1}(x) v(x + \epsilon) - v(x) a_{n-1;1}(x - \epsilon) + [a_{n-1;0}(x), u(x)])\}_{ij} \\
&= \frac{1}{\epsilon} \{u(x) a_{n-1;0}(x) v(x) - v(x) a_{n-1;0}(x - \epsilon) u(x - \epsilon) + a_{n-1;-1}(x + \epsilon) v(x) - v(x) a_{n-1;-1}(x - \epsilon)\}_{ij} \\
&= \frac{n}{\epsilon} [a_{n;0}(x) v(x) - v(x) a_{n;0}(x - \epsilon)]_{ij} + \frac{2}{\epsilon n!} [b_{n;0}(x) v(x) - v(x) b_{n;0}(x - \epsilon)]_{ij} \\
&= n \{v_{ij}, \tilde{H}_n\}_1 + \frac{2}{n!} \sum_{k=1}^N \{v_{ij}, H_{n,k}\}_1.
\end{aligned}$$

The theorem is proved till now. □

For readers' convenience, now we will write down the first several Hamiltonian densities explicitly as follows

$$h_{0,k} = \text{Tr Res } C_{kk} = \text{Tr } E_{kk} = 1, \quad (8.44)$$

$$h_{1,k} = \text{Tr Res } C_{kk} \mathcal{L} = \text{Tr}[(1 - \Lambda)^{-1} u E_{kk} - E_{kk} \frac{\Lambda}{1 - \Lambda} u] = u_{kk}, \quad (8.45)$$

$$\bar{h}_{0,k} = \text{Tr Res } \bar{C}_{kk} = \text{Tr } \tilde{\omega}_0 E_{kk} \tilde{\omega}_0^{-1}, \quad (8.46)$$

$$\bar{h}_{1,k} = \text{Tr Res } \bar{C}_{kk} \mathcal{L} = \text{Tr}(\tilde{\omega}_1 E_{kk} \tilde{\omega}_0^{-1} - \tilde{\omega}_0 E_{kk} \tilde{\omega}_0^{-1} (x - \epsilon) \tilde{\omega}_1 (x - \epsilon) \tilde{\omega}_0^{-1}), \quad (8.47)$$

$$\tilde{h}_0 = 2 \text{Tr Res } \log \mathcal{L} = -\text{Tr } \tilde{\omega}_{0x} \tilde{\omega}_0^{-1}, \quad (8.48)$$

$$\begin{aligned}
\tilde{h}_1 &= \text{Tr Res } [\mathcal{L}(\log \mathcal{L} - 1)] \\
&= \text{Tr} [u - (1 - \Lambda)^{-1} u_x - u \tilde{\omega}_{0x} \tilde{\omega}_0^{-1} - v \tilde{\omega}_{0x} (x - \epsilon) \tilde{\omega}_0^{-1} - \tilde{\omega}_{0x} (x - \epsilon) \tilde{\omega}_0^{-1} (x - \epsilon) \tilde{\omega}_1 (x - \epsilon) \tilde{\omega}_0^{-1}],
\end{aligned} \quad (8.49)$$

with

$$\tilde{\omega}_0 = v \tilde{\omega}_0(x - \epsilon), \quad \tilde{\omega}_1 = u \tilde{\omega}_0 + v \tilde{\omega}_1(x - \epsilon). \quad (8.50)$$

When $N = 1$, the above conserved densities will be the ones of the extended Toda hierarchy in [6]. Similarly as [6], the tau symmetry of the METH can be proved in the following theorem.

Theorem 6. *The Hamiltonian densities of the EMTH have the following tau-symmetry property:*

$$\frac{\partial h_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial h_{j,k}}{\partial t_{\alpha,m}}, \quad \frac{\partial \bar{h}_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial h_{j,k}}{\partial \bar{t}_{\alpha,m}}, \quad (8.51)$$

$$\frac{\partial h_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial \bar{h}_{j,k}}{\partial t_{\alpha,m}}, \quad \frac{\partial \bar{h}_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial \bar{h}_{j,k}}{\partial t_{\alpha,m}}, \quad (8.52)$$

$$\frac{\partial \tilde{h}_m}{\partial t_{j,k}} = \frac{\partial h_{j,k}}{\partial s_m}, \quad \frac{\partial \tilde{h}_m}{\partial t_{j,k}} = \frac{\partial \bar{h}_{j,k}}{\partial s_m}. \quad (8.53)$$

Proof. Let us prove the theorem for the first equation in eqs.(8.53), other cases can be proved in a similar way

$$\begin{aligned} \frac{\partial \tilde{h}_m}{\partial t_{n,k}} &= \frac{2}{m!} \text{TrRes}[-(C_{kk}\mathcal{L}^n)_-, \mathcal{L}^m(\log \mathcal{L} - c_m)] \\ &= \frac{2}{m!} \text{TrRes}[(\mathcal{L}^m(\log \mathcal{L} - c_m))_+, (C_{kk}\mathcal{L}^n)_-] \\ &= \frac{2}{m!} \text{TrRes}[(\mathcal{L}^m(\log \mathcal{L} - c_m))_+, C_{kk}\mathcal{L}^n] = \frac{\partial h_{n,k}}{\partial s_m}. \end{aligned} \quad (8.54)$$

The theorem is proved. \square

This property justifies the following definition of the tau function for the EMTH:

Definition 3. *The tau function τ of the EMTH can be defined by the following expressions in terms of the densities of the Hamiltonians:*

$$h_{j,n} = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial t_{j,n}}, \quad (8.55)$$

$$\bar{h}_{j,n} = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial \bar{t}_{j,n}}, \quad (8.56)$$

$$h_j = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial s_j}. \quad (8.57)$$

With above two different definitions on tau functions of this hierarchy, some mysterious connections between these two tau functions become an open interesting subject. One comes from the wave function and another comes from Hamiltonians. This is not easy and will be included in our future work.

9 Conclusions and Discussions

In this paper, we constructed a new hierarchy called the EMTH and further extended the Sato theory to this hierarchy including Sato equations, matrix wave operators, Hirota quadratic equations, the existence of the tau function. Similarly as extended Toda hierarchy and extended bigraded Toda hierarchy in Gromov-Witten theory of CP^1 , this hierarchy deserves further studying and exploring because of its potential applications in topological quantum fields and Gromov-Witten theory. Because the matrix hierarchy is one special important noncommutative integrable system, what is applications of the EMTH in noncommutative geometry becomes an interesting subject.

Acknowledgements: Chuanzhong Li is supported by the National Natural Science Foundation of China under Grant No. 11201251, Zhejiang Provincial Natural Science Foundation of China under Grant No. LY12A01007, the Natural Science Foundation of Ningbo under Grant No. 2013A610105. Jingsong He is supported by the National Natural Science Foundation of China under Grant No. 11271210, K.C.Wong Magna Fund in Ningbo University. Chuanzhong Li would like to thank Professor Todor E Milanov for his valuable discussion.

References

- [1] M. Toda, Vibration of a chain with nonlinear interaction. J. Phys. Soc. Jpn. 22(1967), 431-436.
- [2] M. Toda, Nonlinear waves and solitons(Kluwer Academic Publishers, Dordrecht, Holland, 1989).
- [3] K. Ueno, K. Takasaki, Toda lattice hierarchy, In “*Group representations and systems of differential equations*” (Tokyo, 1982), 1-95, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.
- [4] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surveys in differential geometry, 1(1991), 243-310.
- [5] B. A. Dubrovin, Geometry of 2D topological field theories, in *Integrable systems and quantum groups* (Montecatini Terme, 1993), 120-348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.
- [6] G. Carlet, B. Dubrovin, Y. Zhang, The Extended Toda Hierarchy, Moscow Math. J. 4 (2004), 313-332.
- [7] T. Milanov, Hirota quadratic equations for the extended Toda hierarchy, Duke Math. J. 138 (2007), 161-178.
- [8] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A, 39 (2006), 9411-9435.
- [9] C. Z. Li, J. S. He, K. Wu, Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda Hierarchy, J. Math. Phys. 51(2010), 043514.
- [10] S. Aoyama, Y. Kodama, Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy, Commun. Math. Phys. 182(1996), 185-219.
- [11] T. Milanov, H. H. Tseng, The spaces of Laurent polynomials, \mathbb{P}^1 -orbifolds, and integrable hierarchies, Journal für die reine und angewandte Mathematik 622 (2008), 189-235.
- [12] C. Z. Li, Solutions of bigraded Toda hierarchy, J. Phys. A 44(2011), 255201.
- [13] C. Z. Li, J. S. He, Dispersionless bigraded Toda hierarchy and its additional symmetry, Rev. Math. Phys. 24(2012), 1230003.
- [14] C. Z. Li, J. S. He, Y. C. Su, Block type symmetry of bigraded Toda hierarchy, J. Math. Phys. 53(2012), 013517.
- [15] T. Milanov, Y. F. Shen, H. H. Tseng, Gromov-Witten theory of Fano orbifold curves, Gamma integral structures and ADE-Toda Hierarchies, arXiv:1401.5778.
- [16] V. G. Kac and J. W. van de Leur, The n-component KP hierarchy and representation theory, J. Math. Phys. 44 (2003) 3245.
- [17] M. Adler, P. van Moerbeke and P. Vanhaecke, Moment matrices and multi-component KP, with applications to random matrix theory, Commun. Math. Phys. 286(2009), 1-38.
- [18] M. Mañas, L. Martínez Alonso, The multicomponent 2D Toda hierarchy: dispersionless limit, Inverse Problems, 25(2009), 11.

- [19] M. Mañas, L. Martínez Alonso, and C. Álvarez Fernández, The multicomponent 2D Toda hierarchy: discrete flows and string equations, *Inverse Problems*, 25(2009), 065007.
- [20] C. Álvarez Fernández, U. Fidalgo Prieto, M. Mañas, The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems, *Inverse Problems*, 26(2010), 055009.
- [21] C. Álvarez Fernández, U. Fidalgo Prieto, and M. Mañas 2010 Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy, *Advances in Mathematics*, 227(2011), 1451-1525.
- [22] G. Carlet, J. van de Leur, Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of \mathbb{P}^1 orbifolds, arXiv:1304.1632, *J. Phys. A: Math. Theor.* 46(2013), 405205.
- [23] J. S. He, L. Zhang, Y. Cheng and Y. S. Li, Determinant representation of Darboux transformation for the AKNS system, *Sci. China A*, 12(2006), 1867-78.
- [24] C. Z. Li, J. S. He, and K. Porsezian, Rogue waves of the Hirota and the Maxwell-Bloch equation, *Phys. Rev. E* 87(2013), 012913.